Outline

1. Main Results
   - Definition of RMBP
   - Logarithmic Asymptotics of RMBP
   - Exact Asymptotics of RMBP

2. Modulated Branching Processes with Absorbing Barriers
   - Hotspot Traffic

3. Related Phenomena and Models
   - Double Pareto
   - Truncated Power Law and Randomly Stopped Processes

4. Concluding Remarks
Main Results

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Concluding Remarks
Proportional growth is everywhere

1. Branching process
2. Modulated by environmental parameters
3. Often repelled from a small value
Reflected Modulated Branching Processes (RMBP)

- \( \{ J_n \in \mathbb{N} \}_{n > -\infty} \) models the environment dynamics.
- \( \{ B^i_n(j) \in \mathbb{N} \} \) is the number of children of the object \( i \) at time \( n \) when the environment is in state \( j \).
- \( l \) is the lower barrier.

Modulated Branching Processes

For \( Z_0 \in \mathbb{N} \), define

\[ Z_{n+1} = \sum_{i=1}^{Z_n} B^i_n(J_n). \]

Reflected Modulated Branching Processes (RMBP)

For \( l, \Lambda_0 \in \mathbb{N} \),

\[ \Lambda_{n+1} = \max \left( \sum_{i=1}^{\Lambda_n} B^i_n(J_n), l \right). \]
\[ \{ J_n \in \mathbb{N} \}_{n \geq -\infty} \text{ models the environment dynamics.} \]

\[ \{ B_n^i(j) \in \mathbb{N} \} \text{ is the number of children of the object } i \text{ at time } n \text{ when the environment is in state } j. \]

\[ l \text{ is the lower barrier.} \]

\[ Z_{n+1} = \sum_{i=1}^{Z_n} B_n^i(J_n). \]

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4. Concluding Remarks
Before stating the Results

**Notation**

\(\{J_n\}\) - modulating process;
\(
\mu(j) \triangleq \mathbb{E}[B_n^j(j)]\) - replication rate when in state \(j\);
\(\Pi_n = \prod_{i=-n}^{-1} \mu(J_i), \ n \geq 1, \ \Pi_0 = I\) and \(M = \sup_{n \geq 0} \Pi_n\).

**Polynomial Gärtner-Ellis conditions**

1. \(n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] \rightarrow \psi(\alpha)\) as \(n \rightarrow \infty\) for \(|\alpha - \alpha^*| < \varepsilon^*\),
2. \(\psi\) is finite in a neighborhood of \(\alpha^*\) and differentiable at \(\alpha^*\) with \(\psi(\alpha^*) = 0, \ \psi'(\alpha^*) > 0\).

- When \(\{J_n\}\) is i.i.d., \(n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] = \psi(\alpha) = \log \mathbb{E}[\mu(J_{-1})^\alpha]\).
- Condition 1) allows some dependency along \(\{J_n\}\), e.g. functions of Markov chain, but not too much \ldots.
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Notation

\{J_n\} - modulating process;
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Condition 1) allows some dependency along \( \{J_n\} \), e.g. functions of Markov chain, but not too much \( \cdots \).
Main Results

$$\bar{B}_n^i \triangleq \sup_k B_n^i(k), \ M = \sup_{n \geq 0} \Pi_n.$$  

Theorem

If \( \{\Pi_n\} \) satisfies the polynomial Gärtner-Ellis conditions, and

\[ E(\Pi_n)^{\alpha^* + \varepsilon} < \infty, \ E[e^{\theta \bar{B}_n^i}] < \infty \ (\varepsilon, \theta > 0, \ n \geq 1), \]  

then,

\[
\lim_{x \to \infty} \frac{\log P[\Lambda > x]}{\log x} = \lim_{x \to \infty} \frac{\log P[M > x]}{\log x} = -\alpha^*.
\]

If \( \sup_j \mu(j) < 1 \) and \( E[e^{\theta \bar{B}_n^i}] < \infty \ (\theta > 0) \), then, \( P[\Lambda > x] = o(e^{-\xi x}) \) for some \( \xi > 0 \), implying

\[
\lim_{x \to \infty} \frac{\log P[\Lambda > x]}{\log x} = -\infty.
\]
What can we learn from the theorem?

What causes power laws? (expansions and contractions...)

Logarithmically asymptotic equivalence between the tails of $M$ and $\Lambda$.

**Reflected Multiplicative Process (RMP) → $M$**

Define for $n \geq 0$ and $M_0 < \infty$

$$M_{n+1} = \max(M_n \cdot \mu(J_n), l),$$

If $\mathbb{E} \log J_n < 0$, then $M_n \xrightarrow{d} M$, $M = \sup_{n \geq 0} \prod_{i=-n}^{i-1} \mu(J_i)$.

**Compare RMBP & RMP**

![Graph comparing RMBP and RMP](image)
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**Compare RMBP & RMP**

[Graph showing comparison of RMBP and RMP]
Sketch of the proof

General points

- Based on **sample path** arguments.

  Representation lemma:

  \[ \Lambda_n \overset{d}{=} \max_{0 \leq i \leq n} Z_{-i} \rightarrow \max_{i \geq 0} Z_{-i} \overset{d}{=} \Lambda. \]

- Identify the **critical time scale** within which \( \max_{n \geq 0} Z_{-n} \) reaches a big value. For all \( \beta > 0 \),

  \[ \sum_{n > x}^{\infty} P[Z_n^l > x] = o \left( \frac{1}{x^{\beta}} \right) \]

  \[ \Rightarrow P[\Lambda > x] \sim P[\Lambda|_{x}] > x. \]
Sketch of the proof

General points

Based on sample path arguments.

Representation lemma:

$$\Lambda_n \overset{d}{=} \max_{0 \leq i \leq n} Z_i \rightarrow \max_{i \geq 0} Z_i \overset{d}{=} \Lambda.$$ 

Identify the critical time scale within which $\max_{n \geq 0} Z_n$ reaches a big value. For all $\beta > 0$,

$$\sum_{n > x} \mathbb{P}[Z_n^l > x] = o \left( \frac{1}{x^\beta} \right)$$

$$\Rightarrow \mathbb{P}[^\Lambda > x] \sim \mathbb{P}[\Lambda_{\lfloor x \rfloor} > x].$$
Sketch of the proof

General points

- Based on **sample path** arguments.
- Representation lemma:

  \[ \Lambda_n \overset{d}{=} \max_{0 \leq i \leq n} Z_i \rightarrow \max_{i \geq 0} Z_{-i} \overset{d}{=} \Lambda. \]

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  \[ \Rightarrow \mathbb{P}[\Lambda > x] \sim \mathbb{P}[\Lambda_{\lfloor x \rfloor} > x]. \]
Sketch of the proof for the upper bound

Increase the lower barrier to get an upper bound.

Choose $l_x = \lfloor x^\epsilon \rfloor \geq l$, then, $\mathbb{P}[\Lambda^{l_x}_x > x] \geq \mathbb{P}[\Lambda^l > x]$, and

$$\mathbb{P}[\Lambda^l > x] = \mathbb{P}\left[\sup_{j \geq 1} Z^l_{-j} > x\right] \leq \mathbb{P}\left[\Lambda^{l_x}_x > x\right] + \sum_{j > x} \mathbb{P}[Z^l_j > x],$$

now, $l_x$ large $\Rightarrow Z^l_{-x} \approx \prod_{i}^{l_x}$

$$\leq \mathbb{P}\left[\sup_{j \geq 1} \prod_{j} (1 + \epsilon)^j > x^{1-\epsilon}\right] + x\mathbb{P}[\mathcal{B}^{l_x}_x] + \sum_{j > x} \mathbb{P}[Z^l_j > x],$$

where $\mathcal{B}^{l_x}_x = \bigcup_{j \geq l_x} \left\{ \sum_{i=1}^{j} B^i_1(J_1) > j\mu(J_1)(1 + \epsilon) \right\}$, $\prod_{j} = \prod_{i=-1}^{j} \mu(J_i)$. 
Sketch of the proof for the lower bound

How to increase the lower barrier but still obtain a lower bound? Given \( \{J_n\}_{n \geq 0} \), if \( \{\Lambda_{n}^{y_1}\} \) and \( \{\Lambda_{n}^{y_2}\} \) are conditional independent, then,

\[
\Lambda_{n}^{y_1+y_2} \overset{d}{\leq} \Lambda_{n}^{y_1} + \Lambda_{n}^{y_2}.
\] (1)

\[
\mathbb{P}[\Lambda_{n}^{l} > x] \geq \mathbb{P}[\Lambda_{n}^{1} > x] = \frac{y \cdot \mathbb{P}[\Lambda_{n}^{1} > x]}{y} \geq \frac{\mathbb{P}[\sum_{j=1}^{y} \Lambda_{n,j}^{1} > y \cdot x]}{y} \geq \frac{\mathbb{P}[\Lambda_{n}^{y} > yx]}{y}.
\]

Choose \( y = \lfloor x^\delta \rfloor \), \( n = \lfloor x \rfloor \). Then, use similar arguments for the upper bound \( \cdots \)

Many details are in the paper.
Predrag R. Jelenković and Jian Tan
Proportional growth, queueing duality and heavy-tails.
*Technique report (in preparation).*

Predrag R. Jelenković and Jian Tan
Modulated branching processes and power-laws.
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4. Concluding Remarks
Difficult! In the scaling region, barrier grows very slowly…

\{J_n : n \geq 1\} i.i.d; \{S_n = \sum_{i=1}^{n} \log J_i : n \geq 1\} is nonlattice with the ladder height distribution \(G_+\); \(\|G_+\| = \mathbb{P}[S_n \leq 0 \text{ for all } n \geq 1] < 1\).

**Theorem**

*If there exits \(\epsilon > 0\) such that \(\mathbb{E}[\mu(J_1)^{\alpha^*}] = 1\), \(\mathbb{E}[\mu(J_1)^{\alpha}] < \infty\) for \(\alpha^* - \epsilon < \alpha < \alpha^* + \epsilon\), \(\mathbb{E}[\log \mu(J_1)] < 0\), and \(\mathbb{E}\left[e^{\theta \sup_k \{|B^1(k) - \mu(k)|\}}\right] < \infty\) in a neighborhood of the origin, then for any \(\gamma > 0\),

\[
\lim_{l \geq (\log x)^3 + \gamma} \mathbb{P}[\Lambda^l / l > x] x^{\alpha^*} = \frac{1 - \|G_+\|}{\alpha^* \int_{0}^{\infty} u e^{\alpha^* u} G_+(du)}.
\]
Difficult! but, in the scaling region, when barrier grows very slowly...

\[ \{ J_n : n \geq 1 \} \text{ i.i.d.} \]

\[ \text{barrier } l \geq (\log x)^{3+\epsilon} \]
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4. Concluding Remarks
New Model of Hotspot Visitors

Hotspot dynamics for Web sites.
A tells B, C to visit the web, and later B may tell D.

Model

- \( \{J_n\} \) - i.i.d. modulating process; \( A_t \) - triggering events.
- Define stopping time \( P \triangleq \inf\{n > 0 : Z_n^l \leq l\} \) for \( l \); after \( P \) the process is killed/absorbed.
- At time \( t \), Poisson \( A_t \) # of objects are created, and each evolves according to an i.i.d. copy of \( J_P \).
New Model of Hotspot Visitors

Hotspot dynamics for Web sites.
A tells B, C to visit the web, and later B may tell D.

Theorem

Assume that $\mathbb{E}[J_n^\alpha] < \infty$, $\alpha - \epsilon < \alpha^* < \alpha + \epsilon$ and $\mathbb{E}[e^{\theta \tilde{B}_n}] < \infty$ for some $\theta > 0$, then,

$$\lim_{x \to \infty} \frac{\log \mathbb{P}[Z_s > x]}{\log x} = -\alpha^*.$$
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New explanations of **double Pareto**

**Double Pareto - frequent empirical observations**

- Transition from heavy-traffic region to large deviation region
- Multiple time scales resulting in double Pareto

**Example (Jelenković & Lazar(1995), queueing context)**

\[ J_n \equiv J(X(n)), \]
\[ X(n) \text{- Markov chain} \]
\[ p_{12} = 1/5000, \]
\[ p_{21} = 1/10, \]
\[ P[J(1) = 1.2] = 0.5, \]
\[ P[J(1) = 0.6] = 0.5, \]
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Truncated power laws

M(B)P with both lower and upper barriers

Similarly as obtaining truncated geometric distributions in finite buffer queue, e.g., $M/M/1/B$

Randomly stopped multiplicative processes $\Leftrightarrow$ RMP

ladder height representation + Pollaczek-Khintchine formula

Randomly stopped MBP
Truncated power laws
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Randomly stopped MBP
Concluding remarks

- RMBP - a new general model of proportional growth
- Under the general polynomial Gärtner-Ellis conditions,
  \[ \text{RMBP} \Rightarrow \text{power laws} \Rightarrow \text{ubiquitous nature of power law} \]
- Discover the duality:
  additive processes (queueing theory) ⇔ proportional growth

Amusing question

What is more frequent, power law or exponential distributions?

What is more frequent, proportional growth or additive growth?

Questions?
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