Proportional Growth, Modulated Branching Processes and Queueing Duality (I)

E6083: lecture 2

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Outline

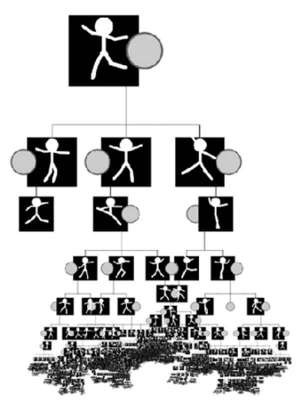
1. Examples and New Model
   - Observations: Proportional growth is everywhere
   - Primary characteristics of all these observations
   - Review of the lognormal distribution

2. RMP and Queueing Duality
   - Connection between queue and RMP
   - Examples

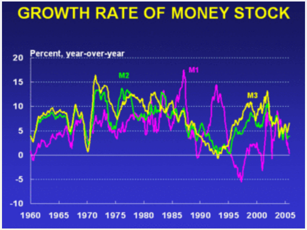
Outline

Proportional growth is everywhere

City population



Wealth growth



Other observations

- The Internet and WWW
- Computer files and Web documents
- Living organisms
- Protein-protein graph, gene regulatory networks
- Firm sizes
- Disease spreading: infectious disease, computer viruses
- Species-area relationships

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Motivation for our new model

Primary characteristics of these observations

1. Replication of independent components
2. Replication is modulated by dynamic environments, causing periods of expansions and contractions , examples $\cdots$
3. Have reflective or porous/absorbing lower barriers, examples $\cdots$

We believe that this model provides a basic structure that explains the origins of power laws.

Motivation for our new model

Primary characteristics of these observations

To capture all these features, we propose new model: modulated branching process with a reflecting or absorbing lower barrier.

This and the following lecture are based on our recent paper

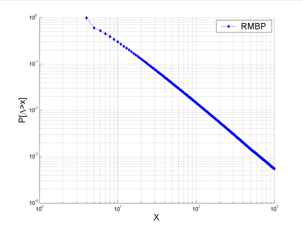
P. R. Jelenkovic and J. Tan, "Modulated branching processes and origins of power laws". In *Proceedings of The Forty-Fourth Annual Allerton Conference on Communication, Control, and Computing*, September 2006.

We believe that this model provides a basic structure that explains the origins of power laws.

Reflected Modulated Branching Processes (RMBP)

Example

Bernoulli process $\{J_n\}_{n\geq 1}$
with $\mathbb{P}[J_n = 1] = 0.4$,
$\mathbb{P}[J_n = 0] = 0.6$;
$\{B_n^i(1)\}_{i\geq 1} \sim \text{Poisson}(1.5)$,
$\{B_n^i(0)\}_{i\geq 1} \sim \text{Poisson}(0.6)$;
$l = 4$;



How do we formulate and analyze this model?

In this class we first study a related model:
reflected multiplicative process (RMP)
which forms the basis to analyze RMBP in the next class.

Reflected Multiplicative Process (RMP)

For $n \geq 0$, $J_n > 0$ and $M_0 < \infty$, define

$$M_{n+1} = \max(M_n \cdot J_n, l),$$

satisfying $\mathbb{E}[\log J_n] < 0$.

Multiplicative Process (MP)

For $n \geq 0$, $J_n > 0$ and $M_0 < \infty$, define

$$M_{n+1} = M_n \cdot J_n,$$

satisfying $\mathbb{V}ar[\log J_n] < \infty$.

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Lognormal Distribution

If X is normal/Gaussian, then e^X is lognormal.

Normal distribution is common: the central limit theorem

Suppose that $\{X_i\}_{i\geq 1}$ is an identically and independently distributed (i.i.d.) random sequence with mean zero and a finite variance σ^2. Then the probability density function $f_n(s)$ of the (normalized) sum of X_i, $S_n = \left(\sum_{i=1}^{n} X_i\right)/n\sqrt{n}$ converges to a normal distribution with unit variance, as n becomes large

$$f_n(s) \to \phi(s) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{s^2}{2}\right).$$

Lognormal distribution

Models of asset dynamics: Binomial lattice model

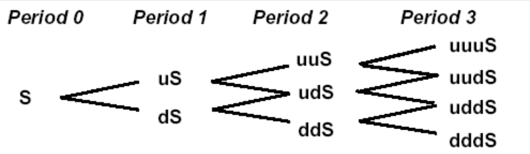


Figure: Binomial lattice model: several periods

- u and d are constants with $u > d$.
- Always positive: lowest value $d^n S$.
- Used for asset pricing (option pricing); modeling demands.
- S_n is approximately lognormal: $S_n = S \times Y_1 \times Y_2 \times \cdots Y_n$.

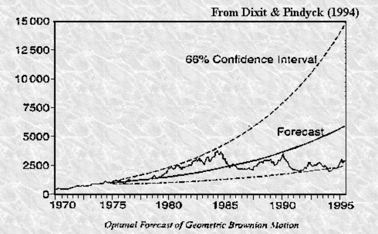
Continuous counterpart

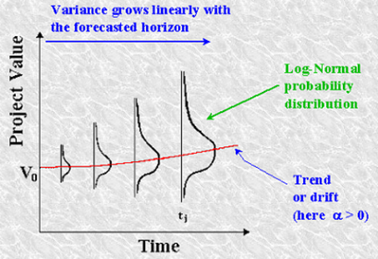
Geometric Brownian Motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Solution: lognormal distribution

$$S(t) = S_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma B_t \right).$$





RMP and duality with queueing

Recursion of RMP

For $l > 0$, $M_0 < \infty$ define RMP for all $n \geq 0$ as

$$M_{n+1} = \max(M_n \cdot J_{n+1}, l).$$

Waiting time of FIFO queue

Let $X_n = \log J_n$, $Q_n = \log M_n$, $l = 1$,

$$Q_{n+1} = \max(Q_n + X_{n+1}, 0).$$

Although, there is a lot of work on specific models of RMPs, it appears that we are the first to make a formal claim the duality between RMPs and queueing.

Outline

The connection between queueing theory and RMP

Assume that $J_n = e^{A_n - C_n}$ where $\{A_i\}, \{C_i\}$ are two mutually independent sequences. Then, $Q_n = \log M_n$ satisfies

$$Q_{n+1} = (Q_n + A_n - C_n)^+.$$

Example

If $\mathbb{P}[C_i > x] = e^{-\mu x}$, $\mathbb{P}[A_i > x] = e^{-\lambda x}$ and $\lambda < \mu$, then Q_n represents the waiting time in a $M/M/1$ queue.

$$\mathbb{P}[Q > x] = \frac{\lambda}{\mu} e^{-(\mu - \lambda)x}, \; x \geq 0,$$

equivalently it yields

$$\mathbb{P}[M > x] = \frac{\lambda}{\mu x^{(\mu - \lambda)}}, \; x \geq 1.$$

The connection between queueing theory and RMP

Assume that $J_n = e^{A_n - C_n}$ where $\{A_i\}, \{C_i\}$ are two mutually independent sequences. Then, $Q_n = \log M_n$ satisfies

$$Q_{n+1} = (Q_n + A_n - C_n)^+.$$

Example

$\mathbb{P}[A_n = 1] = 1 - \mathbb{P}[A_n = 0] = p$,
$\mathbb{P}[C_n = 1] = 1 - \mathbb{P}[C_n = 0] = q$,
then, we have
$\mathbb{P}[Q \geq j] = (1-\rho)\rho^j, j \in \mathbb{N}$ where
$\rho = p(1-q)/q(1-p) < 1$.
Therefore,

$$\frac{1}{x^{\log(1/\rho)}} \leq \mathbb{P}[M \geq x] < \frac{1}{\rho x^{\log(1/\rho)}}.$$

Prior work on reflected multiplicative processes

M. Levy and S. Solomon.
Dynamical Explanation for the Emergence of Power Law in a Stock Market Model.
International Journal of Modern Physics C, 7(1):65-72, 1996.

Moshe Levy and Sorin Solomon.
Power Laws are Logarithmic Boltzmann Laws.
International Journal of Modern Physics C, 7:595-601, 1996.

Rama Cont and Didier Sornette.
Convergent multiplicative processes repelled from zero:power laws and truncated power laws.
Journal de Physique I, 7(3):431-444, 1997.

Reflected Multiplicative Process(RMP)

Xavier Gabaix.
Zipf's Law for Cities: an Explanation.
The Quarterly Journal of Economics, 1999.

A. B. Downey.
The Structural Causes of File Size Distributions.
MASCOTS, 2001.

The connection between queueing theory and RMP generalizes $\cdots$ and implies $\cdots$

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General result on the waiting time of the light-tailed queue

What is light-tailed queue?

For some $\delta > 0$,

$$\mathbb{E}[e^{\delta X_i}] < \infty.$$

Theorem (Logarithmic asymptotics, Glynn & Whitt (1994))

Under very general conditions (Gärtner-Ellis), we have

$$\lim_{x \to \infty} \frac{\log \mathbb{P}[Q > x]}{x} = -\alpha^*.$$

Stability theorem of Loynes (1962) for the $G/G/1$ queue

If the difference X_i of the inter-arrivals and the corresponding service requirements are stationary and ergodic (more general than i.i.d.) and $\mathbb{E}X_i < 0$, then $Q_n \to Q < \infty$. Furthermore, Q has the following explicit representation

$$Q \stackrel{d}{=} \sup_{n \geq 0} S_n$$

where $S_n = \sum_{i=-1}^{-n} X_i$.

Sketch proof of the special case $GI/GI/1$ queue

Upper bound

Use union bound, and then, Chebyshev's inequality

$$\mathbb{P}[Q > x] \leq \sum_{i=1}^{\infty} \mathbb{P}[S_i > x] \leq \sum_{i=1}^{\infty} \frac{\mathbb{E}[e^{\delta S_i}]}{e^{\delta x}}$$

Lower bound

Need find a good n_x, such that the lower bound asymptoticallycoincides with the upper bound.

$$\mathbb{P}[Q > x] \geq \mathbb{P}[S_{n_x} > x]$$

continued... lower bound

Cramér 's theorem

Define $\Lambda(\theta) = \log \mathbb{E}[e^{\theta X_1}]$ and$\Lambda^*(x) = \sup_{\theta \in \mathbb{R}}\{\theta x - \Lambda(\theta)\}$, then,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}[S_n/n \geq y] = -\inf_{u \geq y} \Lambda^*(u).$$

Suppose that $\Lambda^*(u)_{u>y}$ reaches the infimum at θ_y, and,for $x = ny$,

$$\log \mathbb{P}[S_n > x] = \log \mathbb{P}[S_n/n > y] \geq -\theta_y x - n\Lambda(\theta_y) - o(n) \text{ for large } n.$$

Lower bound

For $y > 0$, choose $n = x/y$ (integer part), then,

$$\log \mathbb{P}[Q > x] \geq \log \mathbb{P}[S_{x/y} > x] \geq -\theta_y x - n\Lambda(\theta_y) - o(n)$$
$$= -\frac{x}{y}(\theta_y y - \Lambda(\theta_y)) - o(n) \text{ maximize over } y$$

A general situation: weakly dependent arrivals

Queue : Gärtner-Ellis condition

For queueing, under very general Gärtner-Ellis condition, $\mathbb{P}[\log M > x]$ is "almost" exponential, and therefore,

$$\mathbb{P}[M > x] = \mathbb{P}[\log M > \log x] \approx e^{-\alpha^* \log x} = \frac{1}{x^{-\alpha^*}}.$$

Duality of RMP : Polynomial Gärtner-Ellis conditions

1. $n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] \to \Psi(\alpha)$ as $n \to \infty$ for $\mid \alpha - \alpha^* \mid < \varepsilon^*$, $\Pi_n \triangleq \prod_{i=-1}^{-n} J_i$.
2. Ψ is finite in a neighborhood of α^* and differentiable at α^* with $\Psi(\alpha^*) = 0$, $\Psi'(\alpha^*) > 0$.

Under Polynomial Gärtner-Ellis conditions, RMPs generate power law distributions, which generalizes the existing work.

Outline

RMP: when $\{J_n\}$ is a Markov chain

Example

If $\{J_n\}$ is a Markov chain taking values in $\Sigma = \{u, d\}$, $u \cdot d = 1$, $u > 1$ with transition probabilities $p(d, u) = q = 1 - p(d, d)$, $p(u, d) = p = 1 - p(u, u)$, $p > q$. Let

$$Q_\alpha = \begin{pmatrix} (1-p)u^\alpha & pd^\alpha \\ qu^\alpha & (1-q)d^\alpha \end{pmatrix},$$

then, as $n \to \infty$, $n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] \to \log \mathrm{dev}(Q_\alpha)$, where $\mathrm{dev}(Q)$ is the Perron-Frobenius eigenvalue of matrix Q_α. We have

$$\alpha^* = \frac{\log(1-q) - \log(1-p)}{\log u}. \tag{1}$$

Modulated RMP can result in double Pareto

Example

$\{J_n \equiv J(B(n)\}$ and $B(n)$ is a Markov chain with
$\mathbb{P}[J(1) = 1.2] = 0.5,$
$\mathbb{P}[J(2) = 0.6] = 0.5,$
$\mathbb{P}[J(2) = 1.7] = 0.6,$
$\mathbb{P}[J(2) = 0.25] = 0.4,$
$p_{12} = 1/5000,$
$p_{21} = 1/10.$

This phenomenon was investigated by Jelenkovi'c & Lazar(1995) in the queueing context.