Opinion Dynamics in Social Networks with Stubborn Agents: Equilibrium and Convergence Rate

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Abstract

The process by which new ideas, innovations, and behaviors spread through a large social network can be thought of as a networked interaction game: Each agent obtains information from certain number of agents in his friendship neighborhood, and adapts his idea or behavior to increase his benefit. In this paper, we are interested in how opinions, about a certain topic, form in social networks. We model opinions as continuous scalars ranging from 0 to 1 with 1 (0) representing extremely positive (negative) opinion. Each agent has an initial opinion and incurs some cost depending on the opinions of his neighbors, his initial opinion, and his stubbornness about his initial opinion. Agents iteratively update their opinions based on their own initial opinions and observing the opinions of their neighbors. The iterative update of an agent can be viewed as a myopic cost-minimization response (i.e., the so-called best response) to the others' actions. We study whether an equilibrium can emerge as a result of such local interactions and how such equilibrium possibly depends on the network structure, initial opinions of the agents, and the location of stubborn agents and the extent of their stubbornness. We also study the convergence speed to such equilibrium and characterize the convergence time as a function of aforementioned factors. We also discuss the implications of such results in a few well-known graphs such as Erdos-Renyi random graphs and small-world graphs.

Key words: Multi-agent systems, Markov models, Equilibrium, Eigenvalues

1 Introduction

Rapid expansion of online social networks, such as friendships and information networks, in recent years has raised an interesting question: how do opinions form in a social network? The opinion of each person is influenced by many factors such as his friends, news, political views, area of professional activity, etc. Understanding such interactions and predicting how specific opinions spread throughout social networks has triggered vast research by economists, sociologist, psychologists, physicists, etc.

We consider a social network consisting of n agents. The social network can be modeled as a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ where agents are the vertices and edges indicate pairwise acquaintances. We model opinions as continuous scalars ranging from 0 to 1 with 1(0) representing extremely positive(negative) opinion. For example, such scalers could represent people opinions about the economic situation of the country, ranging from 0 to 1, with an opinion 1 corresponding to perfect satisfaction with the current economy and 0 representing an extremely negative view towards the economy. Agents have some private initial opinions and iteratively update their opinions based on their own initial opinions and observing the opinions of their neighbors. We study whether an equilibrium can emerge as a result of such local interactions and how such equilibrium possibly depends on the graph structure and initial opinions of the agents. In the interaction model, we also incorporate *stubbornness* of agents with respect to their initial opinions and investigate the dependency of the equilibrium on such stubborn agents. Characterizing the convergence rate to the equilibrium as a function of graph structure, location of stubborn agents and their levels of stubbornness is another goal of the current paper.

There has been an interesting line of research trying to explain emergence of new phenomenon, such as spread of innovations and new technologies, based on local inter-

⋆ This research was supported by AFOSR MURI grant FA 9550-10-1-0573 and ARO MURI grant W911NF-12-1-0385. Part of this work has been presented at the 2013 American Control Conference.

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actions among agents, e.g., [5], [6], [22]. Roughly speaking, a coordination game is played between the agents in which adopting a common strategy has a higher payoff and agents behave according to (noisy) best-response dynamics. The references [34], [35] demonstrate how cooperative control problems, e.g. consensus, can be formulated into game-theoretic setting.

There is also a rich and still growing literature on social learning using a Bayesian perspective where individuals observe the actions of others and update their beliefs iteratively about an underlying state variable, e.g., [8], [9], [10]. There is also opinion dynamics based on non-Bayesian models, e.g., those in [1], [2], [3], [7], [11]. In particular, [11] investigates a model in which agents meet and adopt the average of their pre-meeting opinions and there are also forceful agents that influence the opinions of others but may not change their opinions. Under such a model, and assuming that even forceful agents update their opinions when meeting some agents, [11] investigates convergence to the average of the initial opinions and characterizes the amount of divergence from the average due to such forceful agents. As reported in [11], it is significantly more difficult to analyze social networks with several forceful agents that do not change their opinions and requires a different mathematical approach. Our model is closely related to the non-Bayesian framework, this keeps the computations tractable and can characterize the equilibrium in presence of agents that are biased towards their initial opinions (the so-called partially stubborn agents in our paper) or do not change their opinions at all (the so-called fully stubborn agents in our paper). Furthermore, the equilibrium behavior is relevant only if the convergence time is reasonable [6]. Thus, we develop bounds on the rate of convergence that depend on the structure of the social network (such as the diameter of the graph and the relative degrees of stubborn and non-stubborn agents), and the location of stubborn agents and their levels of stubbornness. Based on such bounds, we study the convergence time in social networks with different topologies such as expander graphs, Erdos-Renyi random graphs, and small-world networks. The recent work [12] studies opinion dynamics based on the so-called voter model where each agent holds a binary 0-1 opinion and at each time a randomly chosen agent adopts the opinion of one of its neighbors, and there are also stubborn agents that do not change their states. Under such model, [12] shows that the opinions converge in distribution and characterizes the first and the second moments of this distribution.

When there are no stubborn agents, our model reduces to a *continuous coordination game* where the (noisy) best-response dynamics converge to *consensus* (i.e., a common opinion in which the impact of each agent is directly proportional to its degree in the social network). In this case, the convergence issues are already well understood in the context of consensus and distributed averaging, e.g., [13], [14], [15], [16], [17], [37], [38], [39], [41]. Thus we do not consider this case in this paper.

In this paper, we investigate the convergence issues in presence of stubborn agents. In this case, the opinions do not converge to consensus; however, the opinion of each agent converges to a convex combination of the initial opinions of the stubborn agents. Then our main contributions are the following:

- We exactly characterize the impact of each stubborn agent on such an equilibrium based on appropriately defined hitting probabilities of a random walk over the social network. We also give an interesting electrical network interpretation of the equilibrium.
- Since the exact characterization of convergence time is difficult, we derive appropriate upper-bounds and lower-bounds on the convergence time by extending the frameworks of Diaconis-Stroock [20] and Sinclair [21] to approximate *the largest eigenvalue of sub-stochastic matrices*. In particular, we develop a technique based on completing sub-stochastic matrices to stochastic matrices by adding fictitious stubborn nodes to the social graph.

The organization of the paper is as follows. We start with the definitions and introduce our model in Section 2. Section 3 and 4 contain our main results regarding convergence issues in social networks with stubborn agents. In section 5 we use the results of Section 4 to develop some canonical bounds on the convergence time and discuss the implications of such results in a few well-known graphs. Finally, Section 6 contains our concluding remarks. The proofs of the results are provided in the appendix.

The basic notations used in the paper are as follows. All the vectors are column vectors. x^T denotes the transpose of vector x. A diagonal matrix with elements of vector x as diagonal entries is denoted by $\text{diag}(x)$. x_{max} means the maximum element of vector x. Similarly, x_{min} is the minimum element of vector x. $\mathbb{1}_n$ denotes a vector of all ones of size n. $|S|$ denotes the cardinality of set S. Given two functions f and g, $f = O(g)$ if $\sup_n |f(n)/g(n)| < \infty$. $f = \Omega(g)$ if $g = O(f)$. If both $f = O(g)$ and $f = \Omega(g)$, then $f = \Theta(g)$. We will use the following convenient scalar product and its corresponding norm: given vectors z, y, π in $\mathbb{R}^n$, $\langle z, y\rangle_\pi = \sum_{i=1}^n z_i y_i \pi_i$, and $\|z\|_\pi := \left(\sum_{i=1}^n z_i^2 \pi_i\right)^{1/2}$.

2 Model and definitions

Consider a social network with n agents, denoted by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ where agents are the vertices and edges indicate the pairs of agents that have interactions. For each agent i, define its neighborhood ∂_i as the set of agents that node i interacts with, i.e., $\partial_i := \{j : (i, j) \in \mathcal{E}\}$. Each agent i has an initial opinion $x_i(0) \in [0, 1]$.

Let $x(0) := [x_1(0) \cdots x_n(0)]^T$ denote the vector of initial opinions. We assume each agent i has a cost function of the form

$$J_i(x_i, x_{\partial_i}) = \frac{1}{2} \sum_{j \in \partial_i} (x_i - x_j)^2 + \frac{1}{2} K_i (x_i - x_i(0))^2, \quad (1)$$

that he tries to minimize where $K_i \geq 0$ measures the *stubbornness* of agent i regarding his initial opinion [1]. When none of the agents are stubborn, correspondingly K_i's are all zero, the above formulation defines a *coordination game* with continuous payoffs because any vector of opinions $x = [x_1 \cdots x_n]^T$ with $x_1 = x_2 = \cdots = x_n$ is a *Nash equilibrium* [43]. Here, we consider a synchronous version of the game between the agents. At each time, every agent observes the opinions of his neighbors and updates his opinion based on these observations and also his own initial opinion in order to minimize his cost function. It is easy to check that, for every agent i, the best-response strategy is

$$x_i(t+1) = \frac{1}{d_i + K_i} \sum_{j \in \partial_i} x_j(t) + \frac{K_i}{d_i + K_i} x_i(0), \quad (2)$$

where $d_i = |\partial_i|$ is the degree of node i in graph $\mathcal{G}$. Similar models have been considered in social influence theory, e.g., see [40] where the model assessment is also done by comparing the observed and predicted opinions of groups. Define a matrix $A_{n \times n}$ such that $A_{ij} = \frac{1}{d_i + K_i}$ for $(i,j) \in \mathcal{E}$ and zero otherwise. Also define a diagonal matrix $B_{n \times n}$ with $B_{ii} = \frac{K_i}{d_i + K_i}$ for $1 \leq i \leq n$. Thus, in the matrix form, the best response dynamics are given by

$$x(t+1) = Ax(t) + Bx(0). \quad (3)$$

Iterating (3) shows that the vector of opinions at each time $t \geq 0$ is

$$x(t) = A^t x(0) + \sum_{s=0}^{t-1} A^s B x(0). \quad (4)$$

In the rest of the paper, we investigate the existence of equilibrium, $x(\infty) := \lim_{t \to \infty} x(t)$, under the dynamics (3) in different social networks, with stubborn agents. The equilibrium behavior is relevant only if the convergence time is reasonable [6]. Thus we also characterize the convergence time of the dynamics, i.e., the amount of time that it takes for the agents' opinions to get close to the equilibrium. To be specific, we investigate the convergence issues under the following assumption.

[1] Although we have considered uniform weights for the neighbors, the results in the paper hold under a more general setting when each agent puts a weight w_{ij} for his neighbor j.

Assumption 1 *(i) $\mathcal{G}$ is an undirected connected graph (otherwise, we can consider opinion dynamics separately over each connected subgraph). (ii) At least one agent is stubborn, i.e., $K_i > 0$ for at least one $i \in \mathcal{V}$ (otherwise, it is well known that the dynamics in (2) converge to consensus, i.e. $x_i(\infty) = \frac{1}{2|\mathcal{E}|} \sum_{j=1}^n d_j x_j(0)$ for all i).*

3 Existence and characterization of equilibrium

Consider a social network $\mathcal{G}(\mathcal{V}, \mathcal{E})$ under Assumption 1. Then A is an irreducible *sub-stochastic* matrix with the row-sum of at least one row less than one. Let $\rho_1(A) := \max_i |\lambda_i(A)|$ denote the spectral radius of A. It is well-known that $\rho_1(A)$ of a sub-stochastic matrix A is less than one, and hence, $\lim_{t \to \infty} A^t = 0$. Therefore, by the Perron-Ferobenius theorem, the largest eigenvalue should be positive, real $1 > \lambda_1 > 0$ and $\rho_1(A) = \lambda_1$. Hence, in this case, based on (4), the equilibrium exists and is equal to

$$x(\infty) = \sum_{s=0}^{\infty} A^s B x(0) = (I - A)^{-1} B x(0). \quad (5)$$

Therefore, since $B_{ii} = 0$ for all non-stubborn agents i, the initial opinions of non-stubborn agents will vanish eventually and have no effect on the equilibrium (5).

The matrix form (5) does not give any insight on how the equilibrium depends on the graph structure and the stubborn agents. Next, we describe the equilibrium in terms of explicit quantities that depend on the graph structure, location of stubborn agents and their levels of stubbornness.

Let $\mathcal{S} \subseteq \mathcal{V}$ be the set of stubborn agents and $|\mathcal{S}| \geq 1$. Any agent i in $\mathcal{S}$ is either *fully stubborn*, meaning its corresponding $K_i = \infty$, or it is *partially stubborn*, meaning $0 < K_i < \infty$. Hence, $\mathcal{S} = \mathcal{S}_F \cup \mathcal{S}_P$ where $\mathcal{S}_F$ is the set of fully stubborn agents and $\mathcal{S}_P$ is the set of partially stubborn agents [2]. Next, we construct a *weighted graph* $\hat{\mathcal{G}}(\hat{\mathcal{V}}, \hat{\mathcal{E}})$ based on the original social graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and the location of partially stubborn agents $\mathcal{S}_P$ and their levels of stubbornness K_i, $i \in \mathcal{S}_P$ as follows. Assign weight 1 to all the edges of $\mathcal{G}$. Connect a new vertex u_i to each $i \in \mathcal{S}_P$ and assign a weight K_i to the corresponding edge. Let $\hat{\mathcal{V}} := \mathcal{V} \cup \{u_i : i \in \mathcal{S}_P\}$ and $\hat{\mathcal{E}} := \mathcal{E} \cup \{(i, u_i) : i \in \mathcal{S}_P\}$. Also let w_{ij} denote the weight of edge $(i,j) \in \hat{\mathcal{E}}$. Then $\hat{\mathcal{G}}(\hat{\mathcal{V}}, \hat{\mathcal{E}})$ is a weighted graph with weights $w_{ij} = 1$ for all $(i,j) \in \mathcal{E}$ (the edges of $\mathcal{G}$) and $w_{iu_i} = K_i$ for all $i \in \mathcal{S}_P$. Let $u(\mathcal{S}_P) := \{u_i : i \in \mathcal{S}_P\}$.

[2] We need to distinguish between the case $0 < K_i < \infty$ and $K_i = \infty$ for technical reasons; however, as it will become clear later, the conclusions for $K_i = \infty$ are equivalent to those for $K_i < \infty$ if we let $K_i \to \infty$

Define $w_i := \sum_{j:(i,j)\in\hat{\mathcal{E}}} w_{ij}$ as the weighted degree of vertex $i \in \hat{\mathcal{V}}$. It should be clear that

$$w_i = \begin{cases} d_i + K_i & \text{for } i \in \mathcal{S}_P, \\ d_i & \text{for } i \in \mathcal{V}\backslash\mathcal{S}_P, \\ K_j & \text{for } i = u_j, j \in \mathcal{S}_P. \end{cases} \tag{6}$$

Consider the random walk $Y(t)$ over $\hat{\mathcal{G}}$ where the probability of transition from vertex i to vertex j is $P_{ij} = \frac{w_{ij}}{w_i}$. Assume the walk starts from some initial vertex $Y(0) = i \in \mathcal{V}$. For any $j \in \hat{\mathcal{V}}$ define

$$\tau_j := \inf\{t \geq 0 : Y(t) = j\}, \tag{7}$$

as the first hitting time to vertex j. Also define $\tau := \bigwedge_{j\in\mathcal{S}_F\cup u(\mathcal{S}_P)} \tau_j$ as the first time that the random walk hits any of the vertices in $\mathcal{S}_F \cup u(\mathcal{S}_P)$. The following Lemma characterizes the equilibrium. The proof is provided in Appendix A.

Lemma 1 *The best-response dynamics converge to a unique equilibrium where the opinion of each agent is a convex combination of the initial opinions of the stubborn agents. Based on the random walk over the graph $\hat{\mathcal{G}}$,*

$$x_i(\infty) = \sum_{j\in\mathcal{S}_P} \mathbb{P}_i(\tau = \tau_{u_j})x_j(0) + \sum_{j\in\mathcal{S}_F} \mathbb{P}_i(\tau = \tau_j)x_j(0), \tag{8}$$

for all $i \in \mathcal{V}$, where $\mathbb{P}_i(\tau = \tau_k)$, $k \in \mathcal{S}_F \cup u(\mathcal{S}_P)$, is the probability that the random walk hits vertex k first, among vertices in $\mathcal{S}_F \cup u(\mathcal{S}_P)$, given the random walk starts from vertex i.

Note that $\lim_{K_i\to\infty} \mathbb{P}_i(\tau = \tau_{u_i}) = 1$ for any partially stubborn agent $i \in \mathcal{S}_P$. This intuitively makes sense because as an agent i becomes more stubborn, his opinion will get closer to his own opinion and behaves similarly to a fully stubborn agent.

It should be clear that when there is only one stubborn agent or there are multiple stubborn agents with identical initial opinions, eventually the opinion of every agent will converge to the same opinion as the initial opinion of the stubborn agents.

In general, to characterize the equilibrium, one needs to find probabilities $\mathbb{P}_i(\tau = \tau_k)$, $k \in \mathcal{S}_F \cup u(\mathcal{S}_P)$. Such hitting probabilities have an interesting electrical network interpretation (see Chapter 3 of [18]) as follows. Let $\hat{\mathcal{G}}$ be an electrical network where each edge $(i,j) \in \hat{\mathcal{E}}$ has a conductance w_{ij} (or resistance $1/w_{ij}$). Then $\mathbb{P}_i(\tau = \tau_k)$ is the voltage of node i in the electrical network where node $k \in \mathcal{S}_F \cup u(\mathcal{S}_P)$ is a fixed voltage source of 1 volt and nodes $\mathcal{S}_F \cup u(\mathcal{S}_P)\backslash\{k\}$ are grounded (zero voltage). This determines the contribution of the voltage source k where all the other sources are turned off. Now let vertices $\mathcal{S}_F \cup u(\mathcal{S}_P)$ be fixed voltage sources where the voltage of each source $i \in \mathcal{S}_F$ is $x_i(0)$ volts and the voltage of each source $u_j \in u(\mathcal{S}_P)$, $j \in \mathcal{S}_P$, is $x_j(0)$ volts. By the linearity of the electrical networks (the superposition theorem in circuit analysis), the voltage of each node in such an electrical network equals to the sum of the responses caused by each voltage source acting alone, while all other voltage sources are grounded. Therefore, the opinion of agent i, at equilibrium (8), is just the voltage of node i in the electrical network model. We mention the result as the following lemma and will prove it directly in Appendix B.

Lemma 2 *Consider $\mathcal{G}$ as an electrical network where the conductance of each edge is 1 and each stubborn agent i is a voltage source of $x_i(0)$ volts with an internal conductance K_i. Fully stubborn agents are ideal voltage sources with infinite internal conductance (zero internal resistance). Then, under the best-response dynamics, the opinion of each agent at equilibrium is just its voltage in the electrical network.*

We illustrate the use of the above lemma through the following example.

Example 1 *Consider a one-dimensional social graph, where agents are located on integers $1 \leq i \leq n$. Assume nodes 1 and n are stubborn with initial opinions $x_1(0)$ and $x_n(0)$, and stubbornness parameters $K_1 > 0$ and $K_n > 0$. Using the electrical network model, the current is the same over all edges and equal to $I = (x_1(0) - x_n(0))(\frac{1}{K_1} + \frac{1}{K_n} + n - 1)^{-1}$, and thus the voltage of each node i is $v_i = x_1(0) - I(\frac{1}{K_1} + i - 1)$, for $1 \leq i \leq n$. Hence,*

$$x_i(\infty) = (1 - \alpha_i)x_1(0) + \alpha_i x_n(0),$$

where $\alpha_i := \frac{K_1^{-1}+i-1}{K_1^{-1}+K_n^{-1}+n-1}$. As K_1 increases, the final opinion of i will get closer to stubborn agent 1, and as K_n increases, it will get closer to the opinion of agent n.

4 Convergence time

Although we are able to characterize the equilibrium, the equilibrium behavior is relevant only if the convergence time is reasonable [6]. Next, we characterize convergence time in the case that there is at least one stubborn agent. Let $e(t) = x(t) - x(\infty)$ be the error vector. Trivially $e_i(t) = 0$ for all fully stubborn agents $i \in \mathcal{S}_F$, so we focus on $\tilde{e}(t) := [e_i(t) : i \in \mathcal{V}\backslash\mathcal{S}_F]^T$. The convergence to the equilibrium (5) is geometric with a rate equal to largest eigenvalue of A as stated by the following lemma whose proof is provided in Appendix C.

Lemma 3 *Let $\tilde{\pi} = [\frac{w_i}{Z} : i \in \mathcal{V}\backslash\mathcal{S}_F]^T$ for the weights w_i as in (6) and Z be the normalizing constant such that*

$\sum_{i\in\mathcal{V}\backslash\mathcal{S}_F}\tilde{\pi}_i = 1$. *Then,*

$$\|\tilde{e}(t)\|_{\tilde{\pi}} \le (\lambda_A)^t \|\tilde{e}(0)\|_{\tilde{\pi}}, \tag{9}$$

where λ_A is the largest eigenvalue of A. [3]

Defining the convergence time as $\tau(\nu) := \inf\{t \ge 0 : \|\tilde{e}(t)\|_{\tilde{\pi}} \le \nu\}$ for some fixed $\nu > 0$, we have

$$\left(\frac{1}{1-\lambda_A} - 1\right) \le \frac{\tau(\nu)}{\log(\|\tilde{e}(0)\|_{\tilde{\pi}}/\nu)} \le \frac{1}{1-\lambda_A},$$

so $\tau(\nu) = \Theta\left(\frac{1}{1-\lambda_A}\right)$ as n grows. Let $T := \frac{1}{1-\lambda_A}$. With a little abuse of terminology, we also call T the convergence time.

The exact characterization of λ_A in social networks with very large number of users and many stubborn agents is difficult, hence, we will derive appropriate upper-bounds and lower-bounds that depend on the graph structure, the location of stubborn agents and their levels of stubbornness. The techniques used here are similar to geometric bounds in [20], [21], however, careful modification of such bounds is needed as the results in [20], [21] are for the second largest eigenvalue of stochastic matrices whereas here we are dealing with the largest eigenvalue of sub-stochastic matrices.

Consider the weighted graph $\hat{\mathcal{G}}(\hat{\mathcal{V}},\hat{\mathcal{E}})$ as defined in Section 3. A *path* γ_{ij} from a vertex i to another vertex j in $\hat{\mathcal{G}}$ is a collection of *oriented* edges that connect i to j. For any vertex $i \in \mathcal{V}\backslash\mathcal{S}_F$, consider a path γ_i from i to the set $\mathcal{S}_F \cup u(\mathcal{S}_P)$ that does not intersect itself, i.e., $\gamma_i \equiv \gamma_{ij} = \{(i,i_1),(i_1,i_2),\cdots,(i_m,j)\}$ for some $j \in \mathcal{S}_F \cup u(\mathcal{S}_P)$.

Proceeding along similar arguments as in Diaconis-Stroock [20], we get the following bound that yields an upper-bound on the convergence time (see Appendix D for the proof).

Proposition 1 *Consider the weighted graph $\hat{\mathcal{G}}$. Given a set of paths $\{\gamma_i : i \in \mathcal{V}\backslash\mathcal{S}_F\}$, from $\mathcal{V}\backslash\mathcal{S}_F$ to $\mathcal{S}_F \cup u(\mathcal{S}_P)$, let $|\gamma_i|_w := \sum_{(s,t)\in\gamma_i}\frac{1}{w_{st}}$. Then, the convergence time $T \le 2\xi$, where*

$$\xi := \max_{(x,y)\in\hat{\mathcal{E}}} \xi(x,y), \tag{10}$$

and, for each oriented edge $(x,y) \in \hat{\mathcal{E}}$,

$$\xi(x,y) := \sum_{i:\gamma_i \ni (x,y)} w_i |\gamma_i|_w. \tag{11}$$

[3] In Euclidian norm, $\|e(t)\|_2 \le (\lambda_A)^t\sqrt{\frac{w_{max}}{w_{min}}}\|e(0)\|_2$, where $w_{max} := \max_{i\in\mathcal{V}\backslash\mathcal{S}_F} w_i$ and $w_{min} := \min_{i\in\mathcal{V}\backslash\mathcal{S}_F} w_i$.

It is also possible to modify the arguments of Sinclair [21]. This gives a different bound stated in the following lemma.

Proposition 2 *Consider the weighted graph $\hat{\mathcal{G}}(\hat{\mathcal{V}},\hat{\mathcal{E}})$. Given a set of paths $\{\gamma_i : i \in \mathcal{V}\backslash\mathcal{S}_F\}$ from $\mathcal{V}\backslash\mathcal{S}_F$ to $\mathcal{S}_F \cup u(\mathcal{S}_P)$, we have $T \le 2\eta$, where*

$$\eta := \max_{(x,y)\in\hat{\mathcal{E}}} \eta(x,y), \tag{12}$$

and, for each oriented edge $(x,y) \in \hat{\mathcal{E}}$,

$$\eta(x,y) := \frac{1}{w_{xy}} \sum_{i:\gamma_i \ni (x,y)} w_i |\gamma_i|. \tag{13}$$

The above proposition is very similar to the bound reported in [22] for analyzing the convergence time of a two-strategy coordination game with no stubborn agents but differs by a factor of 2. The factor 2 is not important in investigating the order of the convergence time; however, in graphs with finite number of agents, ignoring this factor yields convergence times that are smaller than the actual convergence time. A short proof is provided in Appendix D for the above lemma.

Intuitively, both $\xi(x,y)$ and $\eta(x,y)$ are measures of *congestion* over the edge (x,y) due to paths that pass through (x,y). See [45] for examples of applications of the above bounds in complete and ring graphs and performance comparison with exact numerical values. In general, computing the upper-bound using Proposition 2 seems to be easier than using Proposition 1.

An upper bound on $1-\lambda_A$, and thus a lower-bound on the convergence time T, is given by the following proposition whose proof is provided in Appendix D

Proposition 3 *Consider the weighted graph $\hat{\mathcal{G}}(\hat{\mathcal{V}},\hat{\mathcal{E}})$, then*

$$1 - \lambda_A \le \min_{U\subseteq\mathcal{V}\backslash\mathcal{S}_F} \psi(U;\hat{\mathcal{G}}), \tag{14}$$

where $\psi(U;\hat{\mathcal{G}}) := \frac{\sum_{i\in U, j\notin U} w_{ij}}{\sum_{i\in U} w_i}$. The minimum is achieved for some connected subgraph with vertex set U.

It is worth emphasizing that the above bounds are quite general and hold for social networks with any finite size and any set of stubborn agents.

5 Canonical bounds via shortest paths

In this section, to gain more insight into factors dominating the convergence speed, we apply Propositions 1, 2, and 3 with the special class of *shortest paths* in social networks with large number of agents. Let $\gamma = \{\gamma_i : i \in \mathcal{V}\backslash\mathcal{S}_F\}$ be the set of shortest paths from vertices $\mathcal{V}\backslash\mathcal{S}_F$ to the set $\mathcal{S}_F \cup u(\mathcal{S}_P)$, so, in fact, for each $i \in \mathcal{V}\backslash\mathcal{S}_F$, $\gamma_i = \gamma_{ij}$ for some $j \in \mathcal{S}_F \cup u(\mathcal{S}_P)$. Let $\Gamma_j \subseteq \mathcal{V}\backslash\mathcal{S}_F$ be the set of nodes that are connected to $j \in \mathcal{S}_F \cup u(\mathcal{S}_P)$ via the shortest paths. We use $|\gamma| := \max_{i\in\mathcal{V}\backslash\mathcal{S}_F} |\gamma_i|$ to denote the maximum length of any shortest path and $|\Gamma| := \max_{j\in\mathcal{S}_F\cup u(\mathcal{S}_P)} |\Gamma_j|$ to denote the maximum number of nodes connected to any node in $\mathcal{S}_F \cup u(\mathcal{S}_P)$ via shortest paths.

Using Proposition 2, for each partially stubborn agent $j \in \mathcal{S}_P$,

$$\eta(j, u_j) = \frac{1}{K_j}(K_j + d_j + \sum_{i\in\Gamma_j} d_i|\gamma_i|) \leq 1 + \frac{\hat{d} + |\gamma||\Gamma|\tilde{d}}{K_{min}},$$

where $\tilde{d} := \max_{i\in\mathcal{V}\backslash\mathcal{S}} d_i$ is the maximum degree of non-stubborn agents, $\hat{d} := \max_{i\in\mathcal{S}} d_i$ is the maximum degree of stubborn agents, and $K_{min} := \min_{j\in\mathcal{S}_P} K_j$ is the minimum stubbornness. Hence, the congestion is dominated by some edge (j, u_j), $j \in \mathcal{S}_P$, only if the stubbornness K_j is sufficiently small.

It follows from our construction of shortest paths that all the paths that pass through an edge $(x, y) \in \mathcal{E}$ are connected to the same $j \in \mathcal{S}_F \cup u(\mathcal{S}_P)$, or equivalently to the same stubborn agent. So for each $(x, y) \in \mathcal{E}$,

$$\eta(x, y) = \sum_{i:\gamma_i\ni(x,y)} d_i|\gamma_i| \leq |\gamma|B\tilde{d},$$

where

$$B := \max_{(x,y)\in\mathcal{E}} |\{i : \gamma_i \ni (x, y)\}|, \tag{15}$$

is the *bottleneck constant*, i.e., the maximum number of shortest paths that pass through any edge of the social network. It is clear that $|\Gamma|/\hat{d} \leq B \leq |\Gamma|$ because B is at least equal to the number of paths that pass through an edge directly connected to a stubborn agent. Therefore, for $K_{min} \leq K^* := \frac{\hat{d}+|\gamma||\Gamma|\tilde{d}}{|\gamma|B\tilde{d}-1}$, η is dominated by congestion over some edge (j, u_j), $j \in \mathcal{S}_P$, and in this regime

$$T \leq 2\left(1 + \frac{\hat{d} + |\gamma||\Gamma|\tilde{d}}{K_{min}}\right). \tag{16}$$

For $K_{min} > K^*$, η is dominated by an edge of the social network which is the bottleneck, and in this regime

$$T \leq 2|\gamma|B\tilde{d}. \tag{17}$$

Dependence on $|\gamma|$, in both regimes, intuitively makes sense as it represents the minimum time required to reach any node in the network from stubborn agents. Hence, the convergence time in general depends on the structure of the social network and the location of the stubborn agents and their levels of stubbornness. There is a dichotomy for high and low levels of stubbornness. For high levels of stubbornness, and in the extreme case of fully stubborn agents, the opinion of the stubborn agent is almost fixed and the convergence time is dominated by the the bottleneck edge and the structure of the social network. For low levels of stubbornness, the transient opinion of stubborn agent may deviate a lot from its equilibrium which could deteriorate the speed of convergence. In fact, for very low levels of stubbornness, this could be the main factor in determining the convergence time. It is worth pointing out that adding more fully stubborn agents, with not necessarily equal initial opinions, or increasing the stubbornness of the agents makes the convergence faster.

5.1 Scaling laws in large social networks

In this section, we use the canonical bounds to derive scaling laws for the convergence time as the size of the social network n grows. For any social network, we can consider two cases: (i) There exists no fully stubborn agent, i. e., all the stubborn agents are partially stubborn (ii) At least one of the agents is fully stubborn.

In both cases, the upper-bound on the convergence time is given by (16) and (17) depending on the levels of stubbornness of partially stubborn agents. In case (ii), if all the stubborn agents are fully stubborn, then the upper-bound on the convergence time is given by (17).

To find a simple lower-bound, we consider the set U in (14) to include all the nodes $\mathcal{V}\backslash\mathcal{S}_F$. This gives the following lower-bound

$$T \geq \frac{\sum_{j\in\mathcal{S}_P} K_j + 2|\mathcal{E}| - \sum_{j\in\mathcal{S}_F} d_j}{\sum_{j\in\mathcal{S}_P} K_j + \sum_{j\in\mathcal{S}_F} d_j}. \tag{18}$$

In investigating the scaling laws, the scaling of the number of stubborn agents and their levels of stubbornness with n could play an important role. Here, we study scaling laws in graphs with a fixed number of stubborn agents, with fixed levels of stubbornness, as the total number of agents n in the network grows. Then, in any connected graph $\mathcal{G}$, based on (18), the smallest possible convergence time is $T = \Omega(|\mathcal{E}|)$ in the case (i) which could be as small as $\Omega(n)$, and $T = \Omega(\frac{|\mathcal{E}|}{\sum_{j\in\mathcal{S}_F} d_j})$ in the case (ii) which could be as small as $\Omega(1)$. It is possible to combine the upperbounds (16) and (17) as follows to obtain a looser upper-bound that holds for social networks with any fixed number of (partially/fully) stubborn agents and fixed levels of stubbornness. Let

d_{max} be the maximum degree of the social graph (possibly depending on n). The upper-bounds show that $T = O(|\gamma| n d_{max})$ for $K_{min} < K^*$ (a threshold depending on the structure of the graph) and $T = O(|\gamma| B d_{max})$ otherwise. Recall that B was the bottleneck constant, and obviously $B < n$, implying that $T = O(n|\gamma| d_{max})$, for a fixed number of stubborn agents consisting of any mixture of partially/fully stubborn agents. Furthermore, it should be clear that $|\gamma|$ is at most equal to the *diameter* δ of the graph, hence,

$$T = O(n \delta d_{max}). \tag{19}$$

Dependence on the diameter intuitively makes sense as it represents the minimum time required to reach any node in the network from an arbitrary stubborn agent.

Fastest convergence: It should be intuitively clear that a *star graph* $\mathcal{G}$, in which a stubborn agent is directly connected to $n-1$ non-stubborn agents with no edges between the non-stubborn agents, should have the fastest convergence. In fact, it is easy to check that $K^* = \Theta(n)$, hence, if the stubborn agent is partially stubborn (case (i)), by (16), $T = \Theta(n)$, and if the stubborn agent is fully stubborn (case (ii)), by (17), $T = \Theta(1)$, both achieving the smallest possible lower-bounds.

Complete graph and ring graph: In the complete graph, with a fixed number of stubborn agents, $\tilde{d} = \hat{d} = n-1$, $|\Gamma| = \Theta(n)$, $|\gamma| = 2$, $B = 1$, and $K^* = \Theta(n)$. Hence, if at least one of the agents is partially stubborn, by (16) and (18), $T = \Theta(n^2)$. If all the stubborn agents are fully stubborn, by (17) and (18), $T = \Theta(n)$.

In the ring network, $\tilde{d} = \hat{d} = 2$, $|\Gamma| = \Theta(n)$, $|\gamma| = \Theta(n)$, $B = \Theta(n)$, and $K^* = \Theta(1)$. Thus $T = O(n^2)$ and $\Omega(n)$ in both cases (i) and (ii).

None of the graphs always has a faster convergence than the other one. For example, in the case of one stubborn agent with a fixed K_1, and n large enough (larger than a constant depending on the value of K_1), the ring network has a faster convergence than the complete graph, while for any fixed n, and K_1 large enough, the complete graph has a faster convergence than the ring.

Expander graphs and trees: Expanders are graph sequences such that any graph in the sequence has good expansion property, meaning that there exists $\alpha > 0$ (independent of n) such that each subset S of nodes with size $|S| \leq n/2$ has at least $\alpha|S|$ edges to the rest of the network. Expander graphs have found extensive applications in computer science and mathematics (see the survey of [30] for a discussion of several applications). An important class of expanders are d-regular expanders, where each node has a constant degree d. Existence of d-regular expanders, for $d > 2$, was first established in [32] via a probabilistic argument. There are various explicit constructions of d-regular expander graphs, e.g., the Zig Zag construction in [29] or the construction in [31].

Recall the upper-bound (19) when there is a fixed number of (fully/partially) stubborn agents. So, for any bounded degree graph, with maximum degree $d > 2$, and diameter δ, $T = O(n\delta)$. It is easy to see that the diameter of a bounded degree graph, with maximum degree d, is at least $\log_{d-1} n$ (Lemma 4.1, [23]). In fact, for a d-regular tree or a d-regular expander, $\delta = O(\log n)$ [4]. Hence, for these graphs, $T = O(n \log n)$ which is almost as fast as the smallest possible convergence time $\Omega(n)$ when there is at least on partially stubborn agent. When all the stubborn agents are fully stubborn, $T = O(n \log n)$ still holds, by (17) because $B = \Theta(n)$ in any bounded degree graph, but, in this case, the convergence is slow compared to the best possible convergence time $\Omega(1)$.

Erdos-Renyi random graphs: Consider an Erod-Renyi random graph with n nodes where each node is connected to any other node with probability p, i.e., each edge appears independently with probability p. To ensure that the graph is connected, we consider $p = \frac{\lambda \log n}{n}$ for some number $\lambda > 1$ [42]. Assume there are a fixed set of stubborn agents with fixed stubbornness parameters. Using the well-known results, the maximin degree of an Erdos-Renyi random graph is $O(\log n)$ with high probability, i.e., with probability approaching to 1 as n grows [42]. Also we know that the diameter is $O\left(\frac{\log n}{\log np}\right) = O\left(\frac{\log n}{\log(\lambda \log n)}\right)$ with high probability (in fact, the diameter concentrates only on a few distinct values [24]). Hence, using the upper-bound (19) gives $T = O\left(n \frac{\log^2 n}{\log \log n}\right)$ with high probability. This is very close to the best possible convergence time in case (i) but far from the best possible convergence time in case (ii).

Small-world graphs: The previous graph models do not capture many spatial and structural aspects of social networks and, hence, are not realistic models of social networks [23]. Motivated by the small world phenomenon observed by Milgram [25], Strogatz-Watts [26] and Kleinberg [27] proposed models that illustrate how graphs with spatial structure can have small diameters, thus, providing more realistic models of social networks. We consider a variant of these models, proposed in [23], and characterize the convergence time to equilibrium in presence of stubborn agents. We consider

[4] To show the latter, consider the lazy random walk over a d-regular expander graph, i.e., with transition probability matrix $P = \frac{M}{2d} + \frac{I}{2}$ where M is the graph's adjacency matrix. Then, it follows from Cheeger's inequality and the expansion property, that the spectral gap $1 - \lambda_2(P) \geq \frac{\alpha^2}{8d^2}$. Using the relation between the special gap and the diameter $\delta < \frac{\log n}{1-\lambda_2(P)}$ [33], we get $\delta \leq \frac{8d^2}{\alpha^2} \log n$.

two-dimensional graphs for simplicity but results are extendable to the higher dimensional graphs as well.

Start with a social network as a grid $\sqrt{n} \times \sqrt{n}$ of n nodes. Hence, nodes i and j are neighbors if their l_1 distance $\|i-j\| = |x_i - x_j| + |y_i - y_j|$ is equal to 1. It follows from (19) that, in presence of a fixed number of stubborn agents in a bounded degree graph, $T = O(n\delta)$, and in the grid, $\delta = 2\sqrt{n}$ obviously, hence $T = O(n\sqrt{n})$. Note that changing the location of the stubborn agents can change the convergence time only by a constant and does not change the order.

Now assume that each node creates q shortcuts to other nodes in the network. A node i chooses another node j as the destination of the shortcut with probability $\frac{\|i-j\|^{-\alpha}}{\sum_{k\neq i}\|i-k\|^{-\alpha}}$, for some parameter $\alpha > 0$. Parameter α determines the distribution of the shortcuts as large values of α produce mostly local shortcuts and small values of α increase the chance of long-range shortcuts. In particular, $q = 1$ and $\alpha = 0$ recovers the Strogatz-Watts model where the shortcuts are selected uniformly at random. It is shown in [28] that for $\alpha < 2$, the graph is an expander with high probability and hence, using the inequality between the diameter and the spectral gap [33] (see the footnote 4), its diameter is of the order of $O(\log n)$ with high probability. We also need to characterize the maximum degree in such graphs. The following lemma is probably known but we were not able to find a reference for it, hence, we have included a proof for it in our technical report [45] for completeness.

Lemma 4 *Under the small-world network model, $d_{max} = O(\log n)$ with high probability.*

Hence, putting everything together, using the upper-bound (19), we get $T = O(n \log^2 n)$. This differs from the smallest possible convergence time in case (i) by a factor of $\log^2 n$ but far from $\Omega(1)$ in case (ii).

6 Conclusions

We viewed opinion dynamics as a local interaction game over a social network. When there are no stubborn agents, the best-response dynamics converge to a common opinion in which the impact of the initial opinion of each agent is proportional to its degree. In the presence of stubborn agents, agents do not reach a consensus but the dynamics converge to an equilibrium in which the opinion of each agent is a convex combination of the initial opinions of the stubborn agents. The coefficients of such convex combinations are related to appropriately defined hitting probabilities of the random walk over the social network's graph. An alternative interpretation is based on an electrical network model of the social network where, at equilibrium, the opinion of each agent is simply its voltage in the electrical network.

The bounds on the convergence time in the paper can be interpreted in terms of location and stubbornness levels of stubborn agents, and graph properties such as diameter, degrees, and the so-called bottleneck constant (15). The bounds provide relatively tight orders for the convergence time in the case of a fixed number of partially stubborn agents (case (i)) but there is a gap between the lower-bound and the upper-bound when some of the stubborn agents are fully stubborn (case (ii)). Tightening the bounds in case (ii) remains as a future work.

Appendix

A Proof of Lemma 1

The transition probability matrix of the random walk over $\hat{\mathcal{G}}$ is given by

$$P = \begin{bmatrix} \hat{A}_{n\times n} & \hat{B}_{n\times|\mathcal{S}_P|} \\ \mathbf{I}_{|\mathcal{S}_P|} & 0 \end{bmatrix}. \tag{A.1}$$

$\mathbf{I}_{|\mathcal{S}_P|}$ is the identity matrix of size $|\mathcal{S}_P|$, i.e., when the walk reaches u_i, it returns to its corresponding stubborn agent i with probability 1. Nonzero elements of $\hat{A}$ correspond to transitions between vertices of $\mathcal{V}$. Nonzero elements of $\hat{B}$ correspond to transitions from a partially stubborn agent $i \in \mathcal{S}_P$ to u_i. The matrices $\hat{A}$ and A only differ in the rows corresponding to agents $\mathcal{S}_F$ which are all-zero rows in A. Notice that $x_i(t) = x_i(0)$ for all $i \in \mathcal{S}_F$ and $t \geq 0$. Hence, we can focus on the dynamics of $\tilde{x}(t) = [x_i(t) : i \in \mathcal{V}\backslash\mathcal{S}_F]^T$.

Let $\tilde{A}$ be the matrix obtained from $\hat{A}$ (or A) by removing rows and columns corresponding to fully stubborn agents $\mathcal{S}_F$. Let $\hat{A}_{\mathcal{S}_F}$ ($A_{\mathcal{S}_F}$) denote the columns of $\hat{A}$ (A) corresponding to $\mathcal{S}_F$. Let $\tilde{B}$ be the matrix obtained from B by (i) replacing the columns corresponding to fully stubborn agents $\mathcal{S}_F$ with $\hat{A}_{\mathcal{S}_F}$ (or $A_{\mathcal{S}_F}$), (ii) removing rows corresponding to $\mathcal{S}_F$, (iii) removing the columns corresponding to non-stubborn agents (which are all zero columns). Then, we have

$$\tilde{x}(t+1) = \tilde{A}\tilde{x}(t) + \tilde{B}x_{\mathcal{S}}(0), \tag{A.2}$$

where $x_{\mathcal{S}}(0) = [x_i(0) : i \in S]^T$. Note that both A and $\tilde{A}$ have the same largest eigenvalue, i.e., $\lambda_A = \lambda_{\tilde{A}}$. The dynamics (A.2) converge to the equilibrium $\tilde{x}(\infty) = (\mathbf{I} - \tilde{A})^{-1}\tilde{B}x_{\mathcal{S}}(0)$.

For each vertex $i \in \mathcal{V}$, and $j \in \mathcal{S}_F$, let $F_{ij} := \mathbb{P}_i(\tau = \tau_j)$ be the probability that random walk hits j first, among vertices in $\mathcal{S}_F \cup u(\mathcal{S}_P)$, given the random walk starts from vertex i. Also, for each vertex $i \in \mathcal{V}$, and $u_j \in u(\mathcal{S}_P)$, let $F_{ij} := \mathbb{P}_i(\tau = \tau_{u_j})$ be the probability that random walk hits u_j first, among vertices in $\mathcal{S}_F \cup u(\mathcal{S}_P)$, given the random walk starts from vertex i. Then, we have the following recursive formulas for the F_{ij} probabilities. For every $i \in \mathcal{V}\backslash\mathcal{S}_F$ and every $j \in S_F$,

$$F_{ij} = \hat{A}_{ij} + \sum_{k\in\mathcal{V}\backslash\mathcal{S}_F} \hat{A}_{ik}F_{kj}, \tag{A.3}$$

and for every $i \in \mathcal{V}\backslash\mathcal{S}_F$ and every $j \in \mathcal{S}_P$,

$$F_{ij} = \hat{B}_{ij} + \sum_{k\in\mathcal{V}\backslash\mathcal{S}_F} \hat{A}_{ik}F_{kj}. \tag{A.4}$$

Note that $\tilde{B}$ is $[\hat{B}\hat{A}_{\mathcal{S}_F}]$ without the rows corresponding to $\mathcal{S}_F$. Hence, putting the two equations together in the matrix form, $F = \tilde{B} + \tilde{A}F$ or $F = (\mathbf{I} - \tilde{A})^{-1}\tilde{B}$.

Note that for any $i \in \mathcal{S}_F$, $F_{ii} = 1$ and $x_i(t) = x_i(0)$ at all times $t \geq 0$. Hence, the equilibrium at each node $i \in \mathcal{V}$, is a convex combination of initial opinions of stubborn agents, where $x_i(\infty) = \sum_{j\in\mathcal{S}} F_{ij}x_j(0)$.

B Proof of Lemma 2

Recall graph $\hat{\mathcal{G}}$ with edge weights $\{w_{ij} : (i,j) \in \hat{\mathcal{E}}\}$. By (2), and taking the limit as $t \to \infty$, the equilibrium is the solution to the following set of linear equations

$$x_i(\infty) = \frac{1}{w_i}\sum_{j\in\partial_i} w_{ij}x_j(\infty), \tag{B.1}$$

for each node $i \in \hat{\mathcal{V}}$, with boundary conditions $x_{u_i}(\infty) = x_i(0)$, $i \in \mathcal{S}_P$, and $x_i(\infty) = x_i(0)$ for $i \in \mathcal{S}_F$. Now assume each edge $(i,j) \in \hat{\mathcal{E}}$ has a conductance w_{ij} and vertices $\mathcal{S}_F \cup u(\mathcal{S}_P)$ are voltage sources where the voltage of each source $i \in \mathcal{S}_F$ is $x_i(0)$ volts and the voltage of each source $u_j \in u(\mathcal{S}_P)$, $j \in \mathcal{S}_P$, is $x_j(0)$ volts. Let v_i be the voltage of node i. Kirchhoff's current law states that the total current entering each node must be zero, i.e., for each node $i \in \mathcal{V}\backslash\mathcal{S}_F$, $\sum_{j\in\partial_i} w_{ij}(v_i - v_j) = 0$ or equivalently,

$$w_i v_i = \sum_{j\in\partial_i} w_{ij}v_j \tag{B.2}$$

which, comparing to (B.1), shows that $x_i(\infty) = v_i$. Note that having a fully stubborn agent i, with $K_i = \infty$, corresponds with connecting i to a fixed voltage of $x_i(0)$ volts with an edge of infinite conductance (short circuit). Hence, K_i's can be interpreted as the internal conductance of the voltage sources. A fully stubborn agent i with $K_i = \infty$ corresponds to an ideal voltage source with zero internal resistance.

C Proof of Lemma 3

From the definition of $e(t)$,

$$\begin{aligned} e(t) &= A^t x(0) + \sum_{s=0}^{t-1} A^s Bx(0) - \sum_{s=0}^{\infty} A^s Bx(0) \\ &= A^t x(0) - \sum_{s=t}^{\infty} A^s Bx(0) \\ &= A^t\Big(x(0) - \sum_{s=0}^{\infty} A^s Bx(0)\Big). \end{aligned}$$

Hence $e(t+1) = Ae(t)$. Let λ_A denote the largest eigenvalue of the irreducible sub-stochastic matrix A. Trivially $e_i(t) = 0$ for all fully stubborn agents $i \in \mathcal{S}_F$. Let $\tilde{e}(t) := [e_i(t) : i \in \mathcal{V}\backslash\mathcal{S}_F]^T$ denote the vector of errors without the fully stubborn agents. Then $\tilde{e}(t) = \tilde{A}\tilde{e}(t-1)$ holds, where $\tilde{A}$ is the matrix obtained from A by removing rows and columns corresponding to agents $\mathcal{S}_F$. Note that $\tilde{A}$ and A have the same largest eigenvalue, i.e., $\lambda_A = \lambda_{\tilde{A}}$.

Consider the Markov chain defined by P in (A.1). It is easy to check that P is reversible[5] with respect to a distribution $\pi = [\pi_i = \frac{w_i}{Z} : i \in \hat{\mathcal{V}}]^T$ where w_i is the weighted degree of vertex i, given by (6), and $Z = 2(|\mathcal{E}| + \sum_{i\in\mathcal{S}_P} K_i)$ is the normalizing constant. Note that $\pi_i\tilde{A}_{ij} = \pi_j\tilde{A}_{ji}$ holds for all $i,j \in \mathcal{V}\backslash\mathcal{S}_F$. By minor abuse of terminology, we would also call $\tilde{A}$ reversible with respect to the distribution $\tilde{\pi} = \Big[\pi_i/\pi(\tilde{A}) : i \in \mathcal{V}\backslash\mathcal{S}_F\Big]^T$, where $\pi(\tilde{A})$ is the normalization constant. Let $\tilde{D} = \mathrm{diag}(\tilde{\pi})$. Then, using the same trick as in the characterization of eigenvalues of a reversible stochastic matrix, $A^* = \tilde{D}^{1/2}\tilde{A}\tilde{D}^{-1/2}$ is symmetric and has the same (real) eigenvalues as $\tilde{A}$. Moreover A^* is diagonalizable with a set of equal right and left eigenvectors $\theta_1, \cdots, \theta_{n-|\mathcal{S}_F|}$. Correspondingly, if $u_1, \cdots, u_{n-|\mathcal{S}_F|}$ denote the left eigenvectors of $\tilde{A}$ and $v_1, \cdots, v_{n-|\mathcal{S}_F|}$ denote its right eigenvectors, it should hold that $u_i = \tilde{D}v_i$. Also from the orthogonality of $\theta_i's$, we have $\langle u_i, u_j\rangle_{1/\tilde{\pi}} = \delta_{ij}$ and $\langle v_i, v_j\rangle_{\tilde{\pi}} = \delta_{ij}$. Using $\{v_1, \cdots, v_{n-|\mathcal{S}_F|}\}$ as a base for $\mathbb{R}^{n-|\mathcal{S}_F|}$, $\tilde{e}(t)$ can be expressed as $\tilde{e}(t) = \sum_{i=1}^{n-|\mathcal{S}_F|}\langle\tilde{e}(t), v_i\rangle_{\tilde{\pi}}v_i$, so $\tilde{A}\tilde{e}(t) = \sum_{i=1}^{n-|\mathcal{S}_F|}\lambda_i\langle\tilde{e}(t), v_i\rangle_{\tilde{\pi}}v_i$. Therefore,

$$\begin{aligned} \|\tilde{e}(t+1)\|_{\tilde{\pi}}^2 &= \sum_i \lambda_i^2\langle\tilde{e}(t), v_i\rangle_{\tilde{\pi}}^2\|v_i\|_{\tilde{\pi}}^2 = \sum_i \lambda_i^2\langle\tilde{e}(t), v_i\rangle_{\tilde{\pi}}^2 \\ &\leq \lambda_A^2\sum_i\langle\tilde{e}(t), v_i\rangle_{\tilde{\pi}}^2 = \lambda_A^2\|\tilde{e}(t)\|_{\tilde{\pi}}^2. \end{aligned}$$

[5] By definition of reversibility, $\pi_i P_{ij} = \pi_j P_{ji}$ for all $i,j \in \hat{\mathcal{V}}$

So $\|\tilde{e}(t+1)\|_{\tilde{\pi}} \leq \lambda_A \|\tilde{e}(t)\|_{\tilde{\pi}}$. Accordingly, $\|\tilde{e}(t)\|_{\tilde{\pi}} \leq \lambda_A^t \|\tilde{e}(0)\|_{\tilde{\pi}}$.

D Proofs of Propositions 1, 2, and 3

The three Propositions are based on the extremal characterization of the eigenvalues. First, we present an extremal characterization for the largest eigenvalue of a sub-stochastic (and reversible) matrix. Then, we state the proofs of individual propositions.

Recall that A and $\tilde{A}$ have the same largest eigenvalue and $\tilde{A}$ is reversible with respect to $\tilde{\pi} = [\pi_i/\pi(\tilde{A}) : i \in \mathcal{V}\backslash\mathcal{S}_F]^T$ (see Appendix C). Thus, it follows from extremal characterization of eigenvalues [19], [4] that

$$1 - \lambda_A = \inf_{f \neq 0} \frac{\langle (\mathbf{I} - \tilde{A})f, f\rangle_{\tilde{\pi}}}{\langle f, f\rangle_{\tilde{\pi}}},$$

where the infimum is over all functions $f : \mathcal{V}\backslash\mathcal{S}_F \to \mathbb{R}$. The above characterization can also be written as

$$1 - \lambda_A = \inf_{\phi \neq 0} \frac{\langle (\mathbf{I} - P)\phi, \phi\rangle_{\pi}}{\langle \phi, \phi\rangle_{\pi}},$$

where now the infimum is over functions $\phi : \hat{\mathcal{V}} \to \mathbb{R}$, such that $\phi\left(\mathcal{S}_F \cup u(\mathcal{S}_P)\right) = 0$, and P is the random walk (A.1). Then, $\langle (\mathbf{I} - P)\phi, \phi\rangle_{\pi} = \mathcal{E}(\phi, \phi)$ where $\mathcal{E}(\phi, \phi)$ is the Dirichlet form

$$\mathcal{E}(\phi, \phi) = \frac{1}{2} \sum_{i,j \in \hat{\mathcal{V}}} \pi_i P_{ij} (\phi(i) - \phi(j))^2,$$

which, in terms of the edge weights of $\hat{\mathcal{G}}$, is equal to

$$\mathcal{E}(\phi, \phi) = \frac{1}{2w} \sum_{i,j \in \hat{\mathcal{V}}} w_{ij} (\phi(i) - \phi(j))^2,$$

where $w := \sum_{i \in \hat{\mathcal{V}}} w_i$. Similarly,

$$\langle \phi, \phi\rangle_{\pi} = \frac{1}{w} \sum_{i \in \mathcal{V}\backslash\mathcal{S}_F} w_i \phi^2(i).$$

For any vertex $i \in \mathcal{V}\backslash\mathcal{S}_F$, consider a path γ_i from i to the set $\mathcal{S}_F \cup u(\mathcal{S}_P)$ that does not intersect itself, i.e., $\gamma_i = \{(i, i_1), (i_1, i_2), \cdots, (i_m, j)\}$ for some $j \in \mathcal{S}_F \cup u(\mathcal{S}_P)$. Note that in this definition the edges are oriented meaning that we distinguish between (x, y) and (y, x). Then, we can write $\phi(i) = \sum_{(x,y) \in \gamma_i} (\phi(x) - \phi(y))$ because $\phi(y) = 0$ if $y \in \mathcal{S}_F \cup u(\mathcal{S}_P)$.

Proof of Proposition 1 The result follows from the extremal characterization of $1 - \lambda_A$. Note that

$$\begin{aligned}
&\langle \phi, \phi\rangle_{\pi} = \frac{1}{w} \sum_{i \in \mathcal{V}\backslash\mathcal{S}_F} w_i \Big(\sum_{(x,y) \in \gamma_i} (\phi(x) - \phi(y)) \Big)^2 \\
&= \frac{1}{w} \sum_{i \in \mathcal{V}\backslash\mathcal{S}_F} w_i \Big(\sum_{(x,y) \in \gamma_i} \frac{1}{\sqrt{w_{xy}}} \sqrt{w_{xy}} (\phi(x) - \phi(y)) \Big)^2 \\
&\leq \frac{1}{w} \sum_{i \in \mathcal{V}\backslash\mathcal{S}_F} w_i \Big(\sum_{(x,y) \in \gamma_i} \frac{1}{w_{xy}} \Big) \Big(\sum_{(x,y) \in \gamma_i} w_{xy} (\phi(x) - \phi(y))^2 \Big) \\
&= \frac{1}{w} \sum_{i \in \mathcal{V}\backslash\mathcal{S}_F} w_i |\gamma_i|_w \Big(\sum_{(x,y) \in \gamma_i} w_{xy} (\phi(x) - \phi(y))^2 \Big) \\
&= \frac{1}{w} \sum_{x,y \in \hat{\mathcal{V}}} w_{xy} (\phi(x) - \phi(y))^2 \Big(\sum_{i : \gamma_i \ni (x,y)} w_i |\gamma_i|_w \Big) \\
&\leq 2\mathcal{E}(\phi, \phi)\xi.
\end{aligned}$$

This concludes the proof. The first inequality follows from Cauchy-Schwarz inequality and the second one from the definition of ξ.

Proof of Proposition 2 The proof is again based on the extremal characterization. Note that

$$\begin{aligned}
\langle \phi, \phi\rangle_{\pi} &= \frac{1}{w} \sum_{i \in \mathcal{V}\backslash\mathcal{S}_F} w_i \Big(\sum_{(x,y) \in \gamma_i} (\phi(x) - \phi(y)) \Big)^2 \\
&\leq \frac{1}{w} \sum_{i \in \mathcal{V}\backslash\mathcal{S}_F} w_i |\gamma_i| \sum_{(x,y) \in \gamma_i} (\phi(x) - \phi(y))^2 \\
&= \frac{1}{w} \sum_{x,y \in \hat{\mathcal{V}}} (\phi(x) - \phi(y))^2 \sum_{i : \gamma_i \ni (x,y)} w_i |\gamma_i| \\
&= \frac{1}{w} \sum_{x,y \in \hat{\mathcal{V}}} w_{xy} (\phi(x) - \phi(y))^2 \frac{1}{w_{xy}} \sum_{i : \gamma_i \ni (x,y)} w_i |\gamma_i| \\
&\leq 2\mathcal{E}(\phi, \phi)\eta,
\end{aligned}$$

which concludes the proof. Again the first inequality follows from Cauchy-Schwarz inequality and the second one from the definition of η.

Proof of Proposition 3 To find an upper bound on $1 - \lambda_A$, consider indicator functions of the form $\mathbb{1}_U$, $U \subseteq \mathcal{V}\backslash\mathcal{S}_F$, in the extremal characterization of eigenvalues. Then, we have

$$\begin{aligned}
1 - \lambda_A &\leq \frac{\mathcal{E}(\mathbb{1}_U, \mathbb{1}_U)}{\langle \mathbb{1}_U, \mathbb{1}_U\rangle_{\pi}} \\
&= \frac{\sum_{i \in U, j \notin U} w_{ij}}{\sum_{i \in U} w_i} =: \psi(U; \hat{\mathcal{G}})
\end{aligned}$$

And accordingly, $1 - \lambda_A \leq \min_{U \subseteq \mathcal{V}\backslash\mathcal{S}_F} \psi(U; \hat{\mathcal{G}})$. It is easy to see that the minimizing U is the vertex set of a connected subgraph of $\mathcal{G}\backslash\mathcal{S}_F$.

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