

# Resource Allocation and Rate Gains in Practical Full-Duplex Systems

Jelena Marašević<sup>1</sup>, Jin Zhou<sup>1</sup>, Harish Krishnaswamy<sup>1</sup>, Yuan Zhong<sup>2</sup>, Gil Zussman<sup>1</sup>

<sup>1</sup>Electrical Engineering Dept., <sup>2</sup>Industrial Engineering and Operations Research Dept.  
Columbia University

New York, NY, 10027, USA

{jelena@ee., jz2495@, harish@ee., yz2561@, gil@ee.}columbia.edu

## ABSTRACT

Full-duplex communication has the potential to substantially increase the throughput in wireless networks. However, the benefits of full-duplex are still not well understood. In this paper, we characterize the full-duplex rate gains in both single-channel and multi-channel use cases. For the single-channel case, we *quantify the rate gain as a function of the remaining self-interference and SNR values*. We also provide a sufficient condition under which the sum rate on a full-duplex channel is concave in the transmission power levels. Building on these results, we consider the multi-channel case. For that case, we *introduce a new realistic model of a small form-factor (e.g., smartphone) full-duplex transceiver* and demonstrate its accuracy via measurements. We study the problem of jointly allocating power levels to different channels and selecting the frequency of maximum self-interference suppression, where the objective is maximizing the sum rate. We develop a *polynomial time algorithm which is nearly optimal under very mild restrictions*. To reduce the running time, we develop an efficient nearly-optimal algorithm under high SINR approximation. Finally, we demonstrate via numerical evaluation the capacity gains in the different use cases and obtain insights into the impact of the remaining self-interference and wireless channel states on system performance.

## 1. INTRODUCTION

Full-duplex (FD) communication – simultaneous transmission and reception on the same frequency channel – is an emerging technology that holds great promise to substantially improve the throughput in wireless networks. The main challenge hindering the implementation of practical FD devices is high self-interference (SI) caused by signal leakage from the transmitter into the receiver circuit. The SI signal is usually many orders of magnitude higher than the desired signal at the receiver’s input, requiring over 100dB of self-interference cancellation (SIC).<sup>1</sup>

Cancelling SI is a very challenging problem. Even though ideas for different techniques of SIC were proposed more than 10 years ago, only recently practical circuit designs that provide sufficient SIC to be employed in Wi-Fi and cellular networks emerged (see [25] and references therein for an overview). Exciting progress was made in the last few years by various research groups, who demonstrated that a combination of SIC techniques employed both in the analog

<sup>1</sup>Namely, it is required to reduce the SI signal power by  $10^{10}$  times.

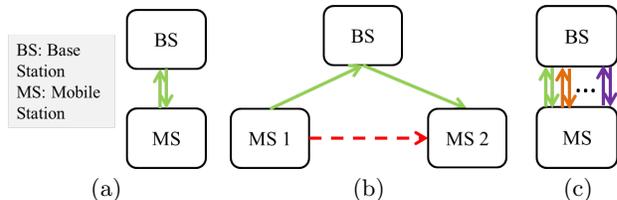


Figure 1: Some possible uses of full-duplex: (a) simultaneous uplink and downlink for one MS; (b) uplink and downlink used by two different MSs and causing inter-node interference (red dashed line), (c) simultaneous uplink and downlink over OFDM channels.

and digital domains can provide SIC levels that can support practical applications [3, 5, 8, 9, 11–14, 17–21, 23, 27, 28].

While there has been significant interest in FD from both industry and academia [2–5, 7–9, 11–15, 17–23, 26–30], *the exact throughput gains resulting from the use of FD are not well understood*. The first implementations of FD transceivers optimistically envisioned 100% rate improvement (e.g., [5, 17]). To achieve such an increase in data rates, the FD receiver would need perfect SIC, namely to cancel SI to at least one order of magnitude below noise to render it negligible. The highest reported SIC [5], however, suppresses the SI to the level of noise in the best case.

Despite this mismatch, much of the work on FD rate improvement assumes perfect SIC [4, 15, 26, 30]. While non-negligible SI has also been considered [2, 7, 22], *there are still no explicit bounds on the rate gains for given FD circuit parameters and parameters of the wireless signal*. Moreover, from a modeling perspective, the frequency selectivity of SIC has not been considered in any analytical work. This is *an important feature that is inherent in conventional compact implementations of an FD transceiver, such as that found in small-form factor mobile devices (e.g., smartphones and tablets)*, where frequency selectivity is mainly a consequence of the cancellation in the RF domain.<sup>2</sup>

Hence, *the main contribution of this paper is a thorough analytical study of rate gains from FD under non-negligible SI*. We consider both *single-channel and multi-channel orthogonal frequency division multiplexing (OFDM) scenarios*. For the multi-channel case, *we develop a new model for frequency-selective SIC in small-form factor devices*. Our results provide explicit guarantees on the rate gains of FD, as a function of receivers’ signal-to-noise ratios (SNRs) and SIC profile. Our analysis provides several insights into the structure of the sum of rates under FD, which we believe will be useful for future work on FD scheduling and MAC

<sup>2</sup>For more details, see [32, 34] and Section 3.

layer algorithm design.

Specifically, we consider three different use cases of FD, as illustrated in Fig. 1: (i) a single channel bidirectional link, where one mobile station (MS) communicates with the base station (BS) both on uplink and downlink (Fig. 1(a)); (ii) two single channel unidirectional links, where one MS communicates with the BS on the uplink, while another MS communicates with the BS on the downlink (Fig. 1(b)); and (iii) a multi-channel bidirectional link, where an MS communicates with the BS over multiple OFDM channels, both on the uplink and downlink (Fig. 1(c)).

For self-interference, we consider two different models. For use case (i) and the BS in all use cases, we model the remaining SI after cancellation as a constant fraction of the transmitted signal. Such design is possible for devices that do not require a very small form factor (e.g., base stations), and was demonstrated in [5].

In the multi-channel case, we rely on the characteristics of transceivers that we recently designed [32,34] and *develop a frequency selective model for the remaining SI in a small form-factor device*. We demonstrate the accuracy of the developed model via measurements with our devices [32,34].

We note that a frequency-selective profile of SIC that we model is inherent to RF cancellers with flat amplitude and phase response (see Section 3). While a mixed-signal SIC architecture [27] does not necessarily have flat amplitude and phase response, we do not consider this architecture because it requires a separate up-conversion path which introduces its own noise and distortion, limiting the resultant RF SIC.

We primarily focus on the problem of maximizing the sum rate under FD. This problem need not be concave or convex in the transmission power levels, since the remaining SI after cancellation often depends on the transmission power level. Due to the lack of good problem structure, existing analytical results (see e.g., [2,7,22]) are often restricted to specialized settings. We obtain several analytical results on the FD rate gains, often under mild restrictions, by examining closely the structural properties of the sum rate function.

In the single-channel cases, we prove that if any rate gain can be achieved from FD, then the gain is maximized by setting the transmission power levels to their respective maximum values. This result is somewhat surprising because of the lack of good structural properties of the sum rate function. We then derive a sufficient condition under which the sum of uplink and downlink rates is concave in both transmission power levels, and show that when this condition is not satisfied, one cannot gain more than 1b/s/Hz (additively) from FD as compared to time-division duplex (TDD). We note that although the model for the remaining SI in the single channel case is relatively simple, it nonetheless captures the main characteristics of the FD sum rate, with some results extending to the multi-channel setting.

In the multi-channel case, we use the frequency-selective SI model for the MS receiver, and study the problem of power allocation and frequency selection, where the objective is to maximize the sum of the rates (in this case, frequency refers to the frequency of maximum SIC of the SI canceller). Although in general it is hard to find an optimal solution to this problem, we develop a nearly optimal algorithm under two mild technical conditions. One condition ensures that the sum rate is concave in transmission powers. This restriction is mild, since we prove that when it does not hold, the possible gains from FD are small. Another condi-

tion imposes bounds on the magnitude of the first derivative of the sum rate in terms of maximum SIC position, and can be easily satisfied in an OFDM system with a large number of channels.

Although the algorithm is nearly optimal and has polynomial time complexity, its running time is relatively high. Therefore, we consider a high SINR approximation of the sum rate, and derive fixed optimal power allocation and maximum SIC frequency setting that maximizes the sum rate up to an additive  $\epsilon$  in time  $O(K \log(1/\epsilon))$ , for any given  $\epsilon$ , where  $K$  is the number of channels.

Finally, we note that throughout the paper, we provide numerical results that quantify the capacity gains in the various use cases and illustrate the impact of different parameters on these gains. For example, for the multi-channel case, we evaluate the rate gains using measured SI of an implemented circuit using both an algorithm for the general SINR regime and for the high SINR regime, and demonstrate that for capacity region extension of over 50% the outputs of the two algorithms match closely.

The rest of the paper is organized as follows. Section 2 reviews related work and Section 3 outlines the challenges in implementing FD transceivers. Section 4 introduces the new model for a small form factor circuit, and the model for the various use cases. Section 5 provides analysis and numerical evaluation for sum rate maximization on a single channel for use case (i). Section 6 analyzes use case (ii) and Section 7 provides analysis, algorithms, and numerical evaluation for use case (iii). We conclude in Section 8.

## 2. RELATED WORK

Possible rate gains from FD have been studied in [2,4,7,15,22,26,30], with much of the work [4,15,26,30] focusing on perfect SIC. Unlike this body of work, we focus on rate gains from FD communication under imperfect SIC.

Non-negligible SI has been considered in [2,7,22]. A sufficient condition for FD gains on a bidirectional link has been provided in [2], for the special case of equal SINRs on uplink and downlink. This model takes into account transmitter's phase noise, which is not considered in this paper. In [2], the FD capacity region is compared to the corresponding TDD capacity region, by normalizing powers so that each station irradiates the same amount of power in both the FD and TDD regimes. In this paper, we provide explicit bounds on FD rate gains as a function of the remaining SI and uplink and downlink SNRs. Unlike [2], we bound the irradiated power of each station individually, which complies with all current Wi-Fi and cellular standards.

Power allocation over orthogonal bidirectional links was considered in [7] and [22] for MIMO and OFDM systems, respectively. The model used in [7] assumes the same amount of SIC and equal power allocation on all channels, which is a less general model than the one that we consider.

A more detailed model with different SIC over channels was considered in [22]. The model used in [22] does not consider dependence of SIC in terms of canceller frequency (although, unlike our work, it takes into account transmitter's phase noise). Similar to [2], [22] compares the FD and TDD capacity regions of a bidirectional link by assuming equal irradiated powers for the two duplexing methods. Optimal power allocation that maximizes one of the rates when the other is fixed is derived for equal powers across channels, while for the general case of unequal powers, [22] only pro-

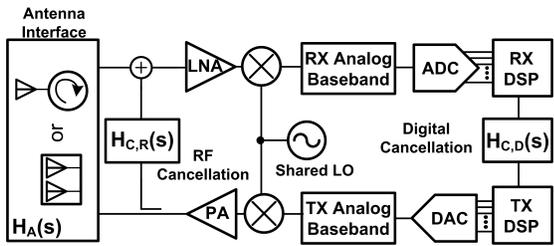


Figure 2: Block diagram of a full-duplex transceiver employing RF and digital cancellation.

vides a suboptimal solution.

Our work relies on structural properties of the sum rate to derive optimal power allocation and maximum SIC frequency setting that maximizes the sum rate. While the model we consider is different than [2,22], we provide a more specific characterization of achievable rate gains, and derive results that provide insights into the rate dependence on the power allocation that allow us to solve a very general problem of rate maximization.

### 3. FD IMPLEMENTATION CHALLENGES

Fig. 2 shows the block diagram of a full-duplex transceiver. There are two antenna interfaces that are typically considered for full-duplex operation: (a) a pair of antennas and (b) a circulator. The advantage of using a circulator is that it allows a single antenna to be shared between the TX and the RX. SI cancellation must be performed in both RF and digital domains to achieve in excess of 100dB SI suppression. The RF canceller taps a reference signal at the output of the power amplifier (PA) and performs SI cancellation at the input of the low-noise amplifier (LNA) at the RX side [10].

Typically, 20-30dB of SI cancellation is required from the RF, given that the antenna interface typically has a TX/RX isolation of 20-30dB [1]. Thus, an overall 50-60dB RF TX/RX isolation is achieved before digital SI cancellation is engaged. This amount of RF TX/RX isolation is critical to alleviate the RX linearity and the ADC dynamic range requirements [10,25]. Digital cancellation further cancels the linear SI as well as the non-linear distortion products generated by the RX or the RF canceller.

A mixed-signal SIC architecture has been proposed in [27], where the digital TX signal is processed and upconverted to RF for cancellation. However, this requires a separate up-conversion path which introduces its own noise and distortion. Moreover, the noise and distortion of the TX analog and RF circuits (such as the power amplifier) are not readily captured in the cancellation signal, limiting the resultant RF SIC. In addition, the dedicated up-conversion path results in area and power overhead. Because of these reasons, we are not considering this SIC architecture in this paper.

For wideband SIC, the transfer function of the canceller must closely track that of the antenna interface across frequency. However, the inherent frequency selectivity of the antenna interface together with selective multi-path-ridden SI channels render this challenging for the RF canceller in particular. The frequency selectivity manifests itself in an antenna interface isolation amplitude and phase response that vary significantly with frequency. A rapidly-varying phase response is representative of a large group delay<sup>3</sup>, re-

<sup>3</sup>The group delay at a certain frequency is related to the rate

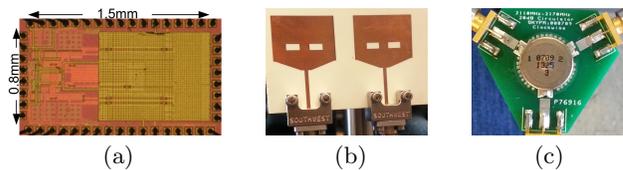


Figure 3: (a) RFIC receiver with RF SI cancellation [32,34] and the two antenna interfaces used in the measurements: (b) an antenna pair and (c) a circulator.

quiring bulky delay lines to replicate the selectivity in the RF canceller [5,10].

The fundamental challenge associated with wideband SI cancellation at RF in a compact form-factor and/or using integrated circuits is the generation of large time delays. The amount of true time delay is linearly proportional to the dimension of the delay structure and inversely proportional to the wave velocity in the medium. To generate 1ns delay in a silicon integrated circuit, a transmission line of 15cm length is required as the relative dielectric constant of silicon oxide is 4. Therefore, a conventional integrated RF SI canceller with dimensions less than 1mm<sup>2</sup> will exhibit negligible delay, and the amplitude and phase response of the canceller can be assumed to be flat with respect to frequency when compared with antenna interface's isolation, limiting the cancellation bandwidth [25,32].

While achieving wideband RF SI cancellation using innovative RFIC techniques is an active research topic (e.g., frequency domain equalization based RF SI cancellation in [33]), in this paper we focus on compact flat amplitude- and phase-based RF cancellers, such as the one implemented in the RFIC depicted in Fig. 3(a) [32,34].

The measured isolation amplitude and group delay response of (a) a PCB antenna pair and (b) a commercial 2110-2170MHz miniature circulator from Skyworks [1] are shown in Fig. 4(a) and Fig. 4(b), respectively. The resultant TX/RX isolations using an RF canceller with flat amplitude and phase response after the antenna interfaces (a) and (b) are shown in Fig. 4(c) and Fig. 4(d), respectively. As Fig. 4(c) and Fig. 4(d) suggest, for -60dB TX/RX isolation after RF cancellation, the bandwidths are about 4MHz and 2.5MHz, respectively.

### 4. MODEL

We consider three uses cases of FD: (i) a bidirectional link, where one mobile station (MS) communicates with the base station (BS) both on uplink and downlink (Fig. 1(a)), (ii) two unidirectional links, where one MS is communicating with the BS on uplink, while another MS is communicating with the BS on downlink (Fig. 1(b)), and (iii) multiple orthogonal bidirectional links (Fig. 1(c)). Note that in (ii) only the BS is operating in FD.

For the multi-channel FD (use case (iii)), we assume that the network bandwidth of size  $B$  is subdivided into  $K$  orthogonal frequency channels of width  $B/K$  each, and index the frequency channels with  $k \in \{1, \dots, K\}$ . An example of such sub-channelization is OFDM with each frequency channel consisting of an integral number of subcarriers.

For all notation that relates to the BS we will use  $b$  in the of change of phase with frequency, and may be interpreted as the time delay experienced by signals at that frequency in passing through the circuit or system.

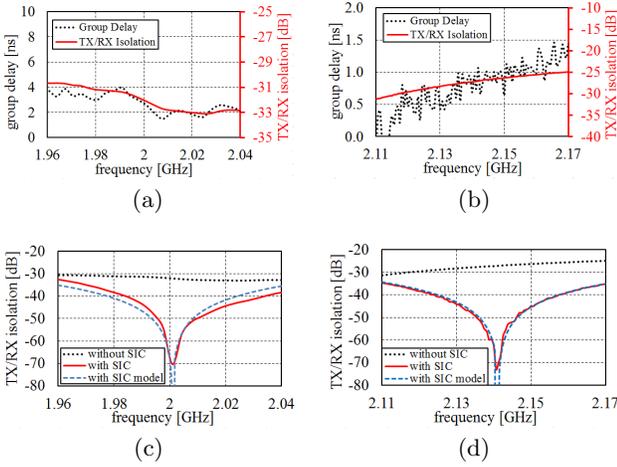


Figure 4: Measured isolation amplitude and group delay of (a) a PCB antenna pair and (b) a commercial 2110-2170 MHz miniature circulator from Skyworks [1], and the resultant TX/RX isolation using an integrated RF canceller with flat amplitude and phase response [32,34] with (c) the antenna pair and (d) the circulator.

subscript. For the notation that relates to the MS in use cases (i) and (iii) we will use  $m$  in the subscript, while in the use case (ii) we will use  $m_1$  and  $m_2$  to refer to MS 1 and MS 2, respectively. Summary of the main notation is provided in Table 1.

The transmission power of a station  $u \in \{b, m, m_1, m_2\}$  on channel  $k$  is denoted by  $P_{u,k}$ , where  $k \in \{1, \dots, K\}$ . In use cases (i) and (ii),  $k$  is omitted from the subscript, since we are considering a single channel.

## 4.1 Remaining SI

**Single-channel FD.** For the single-channel FD, we assume that the remaining SI both at the BS and at an MS can be expressed as a constant fraction of the transmitted power. In particular, if the BS transmits at the power level  $P_b$ , the remaining SI is  $RSI_b = g_b P_b$ , where  $g_b$  is a constant determined by the hardware. Similarly, if an MS transmits at the power level  $P_m$ , its remaining SI is  $RSI_m = g_m P_m$ .

**Multi-channel FD.** We assume that the FD transceiver at the BS has frequency-flat SIC profile, meaning that the remaining SI at the BS on channel  $k$  is  $RSI_{b,k} = g_b P_{b,k}$ , where  $g_b$  is a constant. We note that such FD transceiver design is possible to implement in devices that do not require small form factor of the circuit (e.g., a BS or an access point (AP)), and has been reported [5].

In the rest of this section, we describe the mathematical model of the remaining SI for a small form factor device (MS). We consider a compact/RFIC FD transceiver with a circulator at the antenna interface, described in Section 3, and assume a frequency-flat amplitude and phase response of the canceller, denoted by  $|H_{C,R}|$  and  $\angle H_{C,R}$ , respectively. Amplitude and phase response of the canceller are assumed to be programmable but constant with frequency.

For the antenna interface's TX/RX isolation, we assume a flat amplitude response  $|H_A(f)| = \text{const.} = |H_A|$  and a constant group delay equal to  $\tau$ , so that  $H_A(f) = |H_A|e^{-j2\pi f\tau}$  (measured amplitude and phase response are shown in Fig. 4(b)). For the digital SIC, denoted by  $SIC_D$ , we assume that the amount of cancellation is constant across frequency, as delay

Table 1: Nomenclature.

$m$	Subscript notation for an MS
$b$	Subscript notation for the BS
$K$	Total number of OFDM channels
$k$	Channel index, $k \in \{1, \dots, K\}$
$u, v$	Station indexes, $u, v \in \{b, m, m_1, m_2\}$
$P_{u,k}$	Transmission power of station $u$ on channel $k$
$\bar{P}_u$	Maximum total power: $\sum_{k=1}^K P_{u,k} \leq \bar{P}_u$
$g_b$	Remaining SI at the BS per unit transmitted power
$c$	Position of the maximum SIC frequency, $c \in \mathbb{R}$
$g_m$	Remaining SI at an MS per unit transmitted power for $K = 1$ ; Remaining SI at an MS per unit transmitted power at unit distance from the maximum SIC frequency for $K > 1$
$h_{uv,k}$	Wireless channel gain for signal from $u$ to $v$ on channel $k$ , for $u \neq v$
$N_u$	Thermal noise at station $u$
$\gamma_{uv,k}$	SNR of signal from $u$ to $v$ on channel $k$ , where $u \neq v$
$\gamma_{uv,k}^{\text{XINR}}$	XINR at station $u$ , channel $k$
$\gamma_{uv,k}^{\text{max}}$	$\gamma_{uv,k}$ for $P_u = \bar{P}_u$
$r_k$	Total rate on channel $k$
$r$	Sum rate: $r = \sum_{k=1}^K r_k$
$r_{\text{TDD}}^{\text{max}}$	Maximum TDD rate.

can be easily generated in the digital domain.

Let  $f_k$  denote the central frequency of the  $k^{\text{th}}$  channel, so that  $f_k = f_1 + (k-1)B/K$ . Then the remaining SI after cancellation can be written as:  $RSI_{m,k} = (|H_A|^2 P_{m,k} + |H_{C,R}|^2 P_{m,k} - 2\sqrt{|H_A|^2 P_{m,k} |H_{C,R}|^2 P_{m,k}} \cdot \cos(\angle H_A(f_k) + \angle H_{C,R})) SIC_D^{-1}$ . We assume a common oscillator for the TX and RX, with the phase noise of the oscillator being good enough so that it does not affect the remaining SI.

The RF canceller's settings can be programmed in the field to adjust the frequency at which peak SIC is achieved [32, 34]. With the amplitude ( $|H_{C,R}|$ ) and the phase ( $\angle H_{C,R}$ ) of the RF canceller set to  $|H_A|$  and  $-\angle H_A(f_{c_i})$ , respectively, peak SIC is achieved at frequency  $f_c$ . Therefore, the total remaining SI at the MS on channel  $k$  can be written as:

$$RSI_{m,k} = 2|H_A|^2 P_{m,k} (1 - \cos(2\pi\tau(f_k - f_c))) SIC_D^{-1},$$

where  $\tau$  is the group delay from the antenna interface with a typical value at the order of 1ns (which agrees with the measured group delay in Fig. 4(b)). Frequency bands used by commercial wireless systems are at most 10s of MHz wide. It follows that  $2\pi\tau(f_k - f_c) \ll 1$ , and using the standard approximation  $\cos(x) \approx 1 - x^2/2$  for  $x \ll 1$ , we further get:  $RSI_{m,k} \approx |H_A|^2 P_{m,k} (2\pi\tau)^2 (f_k - f_c)^2 SIC_D^{-1}$ .

Recalling that  $f_k = f_1 + (k-1)B/K = f_0 + kB/K$  for  $f_0 = f_1 - B/K$ , and writing  $f_c$  as  $f_c = f_0 + cB/K$ , for  $c \in \mathbb{R}$ , we can combine all the constant terms and represent the remaining SI as:

$$RSI_{m,k} = g_m P_{m,k} (k - c)^2, \quad (1)$$

where  $g_m = |H_A|^2 (2\pi\tau)^2 (B/K)^2 SIC_D^{-1}$ . Note that even though in this notation we allow  $c$  to take negative values, we will later show that in any solution that maximizes the sum rate it must be  $c \in (1, K)$  (Lemma 7.3).

Fig. 4(d) shows the TX/RX isolation based on Eq. (1) and based on measurement results. The parameter  $g_m$  in Eq. (1) was determined via a least square estimation. As Fig. 4 shows, our model of the remaining SI closely matches the remaining SI of the measured RFIC FD circuit [32,34].

## 4.2 Sum Rate

The total transmitted power of each station is assumed to be bounded as follows. In use cases (i) and (ii):  $P_b \leq \bar{P}_b$ , and each  $P_m, P_{m_1}, P_{m_2} \leq \bar{P}_m$ . In use case (iii):  $\sum_{k=1}^K P_{u,k} \leq \bar{P}_u$ , where  $u \in \{b, m\}$ ,  $\bar{P}_u > 0$ . Channel gain from station  $u$  to station  $v$  on channel  $k$  is denoted as  $h_{uv,k}$  in use case (iii),  $h_{uv}$  in use cases (i) and (ii). Noise level at station  $u$  is assumed to be equal on each channel and is denoted by  $N_u$ .

For the signal transmitted from  $u$  to  $v$ , where  $u, v \in \{b, m, m_1, m_2\}$ ,  $u \neq v$ , and either  $u = b$  or  $v = b$ , we let  $\gamma_{uv,k} = \frac{h_{uv,k}}{N_v}$  denote signal to noise ratio (SNR) at  $v$  on channel  $k$ . Similarly as before, in use cases (i) and (ii) index  $k$  is omitted from the notation. In the use case (ii),  $\gamma_{m_1 m_2}$  denotes the (inter-node-)interference to noise ratio (INR). Self-interference to noise ratio (XINR) at the BS is denoted by  $\gamma_{bb} = \frac{g_b P_b}{N_b}$  in use cases (i) and (ii),  $\gamma_{bb,k} = \frac{g_b P_{b,k}}{N_b}$  in use case (iii). XINR at the MS is denoted by  $\gamma_{mm} = \frac{g_m P_m}{N_m}$  and  $\gamma_{mm,k} = \frac{g_m (k-c)^2 P_{m,k}}{N_m}$  in use cases (i) and (iii), respectively.

We use Shannon's capacity formula for spectral efficiency, and let  $\log(\cdot)$  denote the base 2 logarithm,  $\ln(\cdot)$  denote the natural logarithm. We use the terms "spectral efficiency" and "rate" interchangeably, as the spectral efficiency on a channel is the rate on that channel normalized by  $B/K$ .<sup>4</sup>

In use case (i), the sum rate on the channel is given as:

$$r = \log\left(1 + \frac{\gamma_{mb}}{1 + \gamma_{bb}}\right) + \log\left(1 + \frac{\gamma_{bm}}{1 + \gamma_{mm}}\right). \quad (2)$$

Observe that  $\frac{\gamma_{mb}}{1 + \gamma_{bb}}$  and  $\frac{\gamma_{bm}}{1 + \gamma_{mm}}$  are signal to interference-plus-noise ratios (SINRs) on uplink and downlink, respectively. We will refer to  $r_m = \log\left(1 + \frac{\gamma_{mb}}{1 + \gamma_{bb}}\right)$  as the uplink rate and  $r_b = \log\left(1 + \frac{\gamma_{bm}}{1 + \gamma_{mm}}\right)$  as the downlink rate.

Similarly as for (i), the sum rate for use case (ii) is:

$$r = \log\left(1 + \frac{\gamma_{m_1 b}}{1 + \gamma_{bb}}\right) + \log\left(1 + \frac{\gamma_{b m_2}}{1 + \gamma_{m_1 m_2}}\right). \quad (3)$$

Finally, in use case (iii), the rate on channel  $k$  is given as:

$$r_k = \log\left(1 + \frac{\gamma_{mb,k}}{1 + \gamma_{bb,k}}\right) + \log\left(1 + \frac{\gamma_{bm,k}}{1 + \gamma_{mm,k}}\right), \quad (4)$$

while the sum rate (on all channels) is  $r = \sum_{k=1}^K r_k$ .

**The objective in all problems considered in this paper is to maximize  $r$  subject to the upper bound on total transmitted power and non-negativity constraints.** In use cases (i) and (ii) the variables are  $P_b$  and  $P_m$ , while in the use case (iii) the variables are  $c$ ,  $P_{b,k}$ , and  $P_{m,k}$ , for  $k \in \{1, \dots, K\}$ .

For the purpose of comparison to TDD systems, we will sometimes also consider TDD rates. A pair of uplink and downlink TDD rates  $r_{\text{TDD},m}$  and  $r_{\text{TDD},b}$  is feasible if  $r_{\text{TDD},m} = a_m \log(1 + \gamma_{mb})$ ,  $r_{\text{TDD},b} = a_b \log(1 + \gamma_{bm})$  for  $a_m, a_b \geq 0$  and  $a_m + a_b \leq 1$ . We will let  $r_{\text{TDD}}^{\max}$  denote the maximum of the sum  $r_{\text{TDD},m} + r_{\text{TDD},b}$ .

## 5. A BIDIRECTIONAL FD LINK

In this section, we derive general properties of the sum rate function for use case (i) (Fig. 1(a)).

First, we show that if it is possible for the FD sum rate to exceed the maximum TDD rate, it is always optimal for

<sup>4</sup>Note that this is true both in single-channel and multi-channel cases, since in the single-channel case  $K = 1$ .

the MS and the BS to transmit at their maximum respective power levels (Lemma 5.1). This result is somewhat surprising, because in general, the FD sum rate function does not have good structural properties, e.g., it need not be convex or concave in the transmission power variables.

Building upon this insight, we quantify the FD rate gains by comparing the FD sum rate to corresponding TDD rates (Section 5.2). More specifically, we define a metric that characterizes by how much the FD capacity region extends the corresponding TDD capacity region, and provide a sufficient condition on the system parameters for rate gains to hold.

Finally, we establish a sufficient condition for the FD sum rate function to be concave in transmission powers (Section 5.3). This condition imposes very mild restrictions on the XINRs at the BS and the MS. Moreover, the established condition extends to the multi-channel scenario (use case (iii)), where it plays a crucial role in deriving a nearly-optimal algorithm for the sum rate maximization.

### 5.1 Power Allocation

**LEMMA 5.1.** *If there exists an FD sum rate  $r$  that is higher than the maximum TDD rate, then  $r$  is maximized for  $P_m = \bar{P}_m$ ,  $P_b = \bar{P}_b$ .*

**PROOF.** From (2), the sum rate can be written as:

$$r = \log\left(1 + \frac{h_{mb} P_m}{N_b + g_b P_b}\right) + \log\left(1 + \frac{h_{bm} P_b}{N_m + g_m P_m}\right)$$

Taking partial derivatives of  $r$  directly does not provide conclusive information about the optimal power levels. Instead, we write  $r$  as an increasing function of another function that is easier to analyze. Specifically:

$$r = \log\left(\left(1 + \frac{h_{mb} P_m}{N_b + g_b P_b}\right) \cdot \left(1 + \frac{h_{bm} P_b}{N_m + g_m P_m}\right)\right) = \log(1 + \gamma), \quad \text{where}$$

$$\gamma = \frac{h_{mb} P_m}{N_b + g_b P_b} + \frac{h_{bm} P_b}{N_m + g_m P_m} + \frac{h_{mb} P_m}{N_b + g_b P_b} \cdot \frac{h_{bm} P_b}{N_m + g_m P_m}.$$

Since  $r$  is strictly increasing in  $\gamma$ , to maximize  $r$  it suffices to determine  $P_m, P_b$  that maximize  $\gamma$ .

The first and the second partial derivative of  $\gamma$  with respect to  $P_m$  are:

$$\frac{\partial \gamma}{\partial P_m} = \frac{h_{mb}}{N_b + g_b P_b} + \frac{h_{bm} P_b}{(N_m + g_m P_m)^2} \left( \frac{h_{mb} N_m}{N_b + g_b P_b} - g_m \right), \quad (5)$$

$$\frac{\partial^2 \gamma}{\partial P_m^2} = -2 \frac{h_{bm} P_b g_m}{(N_m + g_m P_m)^3} \left( \frac{h_{mb} N_m}{N_b + g_b P_b} - g_m \right). \quad (6)$$

From (5) and (6):

1. If  $\frac{h_{mb} N_m}{N_b + g_b P_b} - g_m \geq 0$ , then  $\frac{\partial^2 \gamma}{\partial P_m^2} \leq 0$  and  $\frac{\partial \gamma}{\partial P_m} > 0$ , which implies that  $\gamma$  is concave and strictly increasing in  $P_m$ , and therefore maximized for  $P_m = \bar{P}_m$ .
2. If  $\frac{h_{mb} N_m}{N_b + g_b P_b} - g_m < 0$ , then  $\frac{\partial^2 \gamma}{\partial P_m^2} > 0$ , which means that  $\gamma$  is strictly convex in  $P_m$ . Therefore,  $\gamma$  is maximized at either  $P_m = 0$  or  $P_m = \bar{P}_m$ . Note that if  $P_m = 0$ , there is no signal on uplink, in which case FD rate equals the maximum TDD uplink rate.

Similarly, taking the first and the second partial derivative of  $\gamma$  with respect to  $P_b$ :

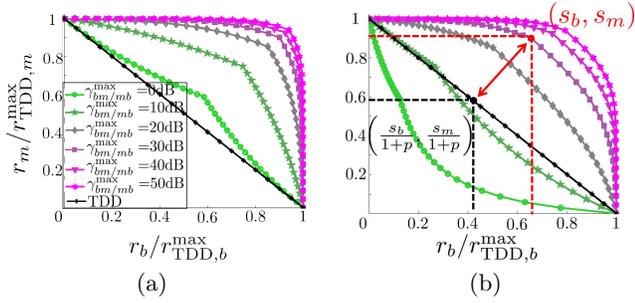


Figure 5: TDD and FD capacity regions, and FD extension. The capacity region is plotted for equal maximum SNRs:  $\gamma_{mb}^{\max} = \gamma_{bm}^{\max} \equiv \gamma_{bm/mb}^{\max}$  and two cases of maximum XINRs: (a)  $\gamma_{bb}^{\max} = 1, \gamma_{mm}^{\max} = 1$  and (b)  $\gamma_{bb}^{\max} = 1, \gamma_{mm}^{\max} = 10$ .

1. If  $\frac{h_{bm}N_b}{N_m + g_m P_m} - g_b \geq 0$ ,  $\gamma$  is concave and strictly increasing in  $P_b$ , and therefore maximized for  $P_b = \bar{P}_b$ .
2. If  $\frac{h_{bm}N_b}{N_m + g_m P_m} - g_b < 0$ ,  $\gamma$  is convex in  $P_b$ , and therefore attains its maximum either at  $P_b = 0$  (FD rate equals maximum TDD downlink rate) or  $P_b = \bar{P}_b$ .

□

## 5.2 Mapping Gain over SINR Regions

In this section we quantify the FD rate gains by comparing the FD capacity region to the corresponding TDD capacity region. Let  $r_b = \log(1 + \frac{\gamma_{bm}}{1 + \gamma_{mm}^{\max}})$ ,  $r_m = \log(1 + \frac{\gamma_{mb}}{1 + \gamma_{bb}^{\max}})$  denote downlink and uplink rates, respectively and let  $r_{TDD,b}^{\max} = \log(1 + \gamma_{bm}^{\max})$ ,  $r_{TDD,m}^{\max} = \log(1 + \gamma_{mb}^{\max})$  denote the maximum TDD rates. The FD capacity region is the set of all points  $(r_b, r_m)$  such that  $P_m \in [0, \bar{P}_m]$ ,  $P_b \in [0, \bar{P}_b]$ , while the TDD capacity region is the convex hull of the points  $(0, 0)$ ,  $(r_{TDD,b}^{\max}, 0)$ , and  $(0, r_{TDD,m}^{\max})$ .

We also let  $s_b = \log(1 + \frac{\gamma_{bm}^{\max}}{1 + \gamma_{mm}^{\max}})$  and  $s_m = \log(1 + \frac{\gamma_{mb}^{\max}}{1 + \gamma_{bb}^{\max}})$  be the FD downlink and uplink rates when both stations transmit at their maximum power levels  $\bar{P}_b, \bar{P}_m$ .

Figure 5 shows FD and TDD capacity regions for symmetric maximum SNRs  $\gamma_{mb}^{\max} = \gamma_{bm}^{\max}$  and two cases of maximum XINRs:  $\gamma_{bb}^{\max} = \gamma_{mm}^{\max} = 1$  and  $\gamma_{bb}^{\max} = 1, \gamma_{mm}^{\max} = 10$ . Here, the axes are normalized by  $r_{TDD,b}^{\max}$  and  $r_{TDD,m}^{\max}$ , respectively. Note that FD capacity regions are not necessarily convex (Fig. 5(b) for  $\gamma_{bm/mb}^{\max} = 0\text{dB}$  and  $\gamma_{bm/mb}^{\max} = 10\text{dB}$ ).

Lemma 5.1 states that the maximizer of the FD sum rate is either  $(r_{TDD,b}^{\max}, 0)$ ,  $(0, r_{TDD,m}^{\max})$  or  $(s_b, s_m)$ . In particular, to see whether FD operation increases the sum rate, it suffices to check whether  $s_b + s_m > \max\{r_{TDD,b}^{\max}, r_{TDD,m}^{\max}\}$ . This motivates us to focus on the pair  $(s_b, s_m)$  when considering by how much the FD capacity region extends the corresponding TDD capacity region. We introduce the following definitions (see Fig. 5(b) for a geometric interpretation).

**Definition 1.** FD extends the corresponding TDD capacity region if the FD rate pair  $(s_b, s_m)$  lies outside the TDD capacity region.

**Definition 2.** FD extends the corresponding TDD capacity region by  $p \cdot 100\%$  if it extends the TDD capacity region and for the rate pair  $(s_b, s_m)$   $p$  is the smallest number for which  $\frac{s_b}{1+p}, \frac{s_m}{1+p}$  is inside the TDD capacity region.

The following lemma provides a necessary and sufficient condition for the capacity region extension of  $p \cdot 100\%$ .

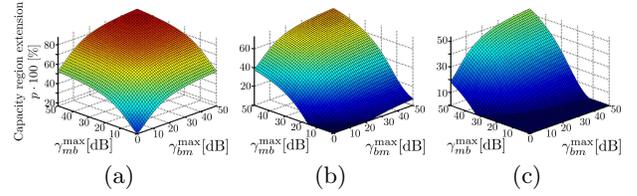


Figure 6: TDD capacity region extension due to FD as a function of SNRs for (a)  $\gamma_{bb}^{\max} = 1, \gamma_{mm}^{\max} = 1$ , (b)  $\gamma_{bb}^{\max} = 1, \gamma_{mm}^{\max} = 10$ , and (c)  $\gamma_{bb}^{\max} = 1, \gamma_{mm}^{\max} = 100$ .

LEMMA 5.2. *FD extends the TDD capacity region by  $p \cdot 100\%$ , where  $p \geq 0$ , if and only if:*

$$\frac{\log\left(1 + \frac{\gamma_{bm}^{\max}}{1 + \gamma_{mm}^{\max}}\right)}{\log(1 + \gamma_{bm}^{\max})} + \frac{\log\left(1 + \frac{\gamma_{mb}^{\max}}{1 + \gamma_{bb}^{\max}}\right)}{\log(1 + \gamma_{mb}^{\max})} = 1 + p. \quad (7)$$

PROOF SKETCH. The proof is based on the fact that since  $p$  is the smallest number for which the point  $(\frac{s_b}{1+p}, \frac{s_m}{1+p})$  is in the TDD capacity region,  $(\frac{s_b}{1+p}, \frac{s_m}{1+p})$  must lie on the line connecting  $r_{TDD,b}^{\max}$  and  $r_{TDD,m}^{\max}$  (Fig. 5(b)), and therefore:

$$\begin{aligned} \frac{s_m}{1+p} &= r_{TDD,m}^{\max} - \frac{r_{TDD,m}^{\max}}{r_{TDD,b}^{\max}} \frac{s_b}{1+p} \\ \Leftrightarrow \frac{\log\left(1 + \frac{\gamma_{bm}^{\max}}{1 + \gamma_{mm}^{\max}}\right)}{\log(1 + \gamma_{bm}^{\max})} + \frac{\log\left(1 + \frac{\gamma_{mb}^{\max}}{1 + \gamma_{bb}^{\max}}\right)}{\log(1 + \gamma_{mb}^{\max})} &= 1 + p. \end{aligned}$$

□

Fig. 6 shows the TDD capacity region extension due to FD operation, as a function of received signals' SNR, for BS FD receiver that cancels SI to the noise level and MS FD receiver that cancels SI to (1) the noise level (Fig. 6(a)), (2) one order of magnitude above noise (Fig. 6(b)), and (3) two orders of magnitude above noise (Fig. 6(b)). Somewhat surprisingly, even in the case when the MS cancels SI to two orders of magnitude above noise, capacity region extension can be as high as 50% for sufficiently high SNRs (Fig. 6(b)).

## 5.3 Sum Rate Concavity

In this section, we establish a sufficient condition for the sum rate to be (strictly) concave, and increasing in  $P_m$  and in  $P_b$  (Condition 5.3). We also show that when the condition fails, we do not get appreciable rate gains from using FD operation, as compared to the maximum rate achievable by TDD operation.

$$\text{CONDITION 5.3. } \gamma_{mm} \leq \frac{\gamma_{mb}}{1 + \gamma_{bb}} \text{ and } \gamma_{bb} \leq \frac{\gamma_{bm}}{1 + \gamma_{mm}}.$$

PROPOSITION 5.4. *If  $\gamma_{mm} \leq \frac{\gamma_{mb}}{1 + \gamma_{bb}}$ , the sum rate  $r$  is strictly concave and strictly increasing in  $P_m$ . Similarly, if  $\gamma_{bb} \leq \frac{\gamma_{bm}}{1 + \gamma_{mm}}$ ,  $r$  is strictly concave and strictly increasing in  $P_b$ . Thus, when Condition 5.3 holds,  $r$  is strictly concave and strictly increasing both in  $P_m$  and in  $P_b$ . Furthermore, when Condition 5.3 does not hold,  $r - r_{TDD}^{\max} < 1b/s/Hz$ .*

PROOF. From the proof of Lemma 5.1, we can express  $r$  as  $r = \log(1 + \gamma)$ , where  $\gamma$  is strictly increasing and concave in  $P_m$  whenever

$$\frac{h_{mb}N_m}{N_b + g_b P_b} - g_m \geq 0. \quad (8)$$

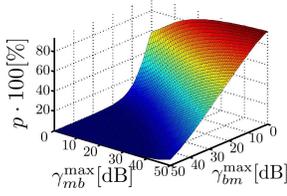


Figure 7: Capacity region extension when Condition 5.3 is not satisfied. Non-negligible gains are possible only when uplink SNR is at least an order of magnitude higher than the downlink SNR, the case that is irrelevant in practice.

Multiplying both sides of (8) by  $\frac{P_m}{N_m}$  and reordering terms:

$$\frac{h_{mb}P_m}{N_b + g_bP_b} \geq \frac{g_mP_m}{N_m} \Leftrightarrow \gamma_{mm} \leq \frac{\gamma_{mb}}{1 + \gamma_{bb}}.$$

Whenever (8), or equivalently, the inequality  $\gamma_{mm} \leq \frac{\gamma_{mb}}{1 + \gamma_{bb}}$ , holds, since  $\gamma > 0$ ,  $\frac{\partial r}{\partial P_m} > 0$ ,  $\frac{\partial^2 r}{\partial P_m^2} \leq 0$ :

$$\begin{aligned} \frac{\partial r}{\partial P_m} &= \frac{1}{1 + \gamma} \cdot \frac{\partial \gamma}{\partial P_m} > 0, \text{ and,} \\ \frac{\partial^2 r}{\partial P_m^2} &= -\frac{1}{(1 + \gamma)^2} \cdot \left( \frac{\partial \gamma}{\partial P_m} \right)^2 + \frac{1}{1 + \gamma} \cdot \frac{\partial^2 \gamma}{\partial P_m^2} < 0, \end{aligned}$$

and therefore  $r$  is strictly increasing and strictly concave in  $P_m$ . Similarly, whenever  $\gamma_{bb} \leq \frac{\gamma_{bm}}{1 + \gamma_{mm}}$ ,  $r$  is strictly increasing and strictly concave in  $P_b$ .

Now suppose that Condition 5.3 does not hold. Then, either  $\gamma_{mm} > \frac{\gamma_{mb}}{1 + \gamma_{bb}}$  or  $\gamma_{bb} > \frac{\gamma_{bm}}{1 + \gamma_{mm}}$ . Suppose that  $\gamma_{mm} > \frac{\gamma_{mb}}{1 + \gamma_{bb}}$ . Then:

$$\begin{aligned} r &= \log \left( 1 + \frac{\gamma_{mb}}{1 + \gamma_{bb}} \right) + \log \left( 1 + \frac{\gamma_{bm}}{1 + \gamma_{mm}} \right) \\ &< \log \left( 1 + \frac{\gamma_{mb}}{1 + \gamma_{bb}} \right) + \log \left( 1 + \frac{\gamma_{bm}}{1 + \frac{\gamma_{mb}}{1 + \gamma_{bb}}} \right) \\ &= \log \left( 1 + \gamma_{bm} + \frac{\gamma_{mb}}{1 + \gamma_{bb}} \right) \\ &= \log \left( 2 \cdot \left( 1 + \frac{1}{2} \left( \gamma_{bm} + \frac{\gamma_{mb}}{1 + \gamma_{bb}} - 1 \right) \right) \right) \\ &= 1 \text{b/s/Hz} + \log \left( 1 + \frac{1}{2} \left( \gamma_{bm} + \frac{\gamma_{mb}}{1 + \gamma_{bb}} - 1 \right) \right). \end{aligned}$$

Since  $\frac{1}{2} \left( \gamma_{bm} + \frac{\gamma_{mb}}{1 + \gamma_{bb}} - 1 \right) < \max\{\gamma_{mb}, \gamma_{bm}\}$ , it follows that  $r < 1 \text{b/s/Hz} + r_{\text{TDD}}^{\max}$ , which completes the proof for  $\gamma_{mm} > \frac{\gamma_{mb}}{1 + \gamma_{bb}}$ . The proof for the case  $\gamma_{bb} > \frac{\gamma_{bm}}{1 + \gamma_{mm}}$  follows the same line of argument and is omitted for brevity.  $\square$

When Condition 5.3 does not hold, the bound  $r - r_{\text{TDD}}^{\max} < 1 \text{b/s/Hz}$  limits the additive increase in FD rate. However, it does not bound the FD capacity region extension. Therefore, we plot an upper bound on the FD capacity region extension in Fig. 7 for a perfect SIC at the BS ( $\gamma_{bb} = 0$ ) and  $\gamma_{mm} = \frac{\gamma_{mb}}{1 + \gamma_{bb}} = \gamma_{mb}$ . As Fig. 7 shows, to have non-negligible capacity region extension, uplink SNR should be higher than the downlink SNR  $\gamma_{bm}$ . This is rarely the case in practice, since in cellular and Wi-Fi networks typically the opposite holds, i.e.,  $\gamma_{bm} > \gamma_{mb}$ , due to better antenna gain and higher transmission power at the BS/AP.

Note that the condition for concavity in  $P_b$ ,  $\gamma_{bb} \leq \frac{\gamma_{bm}}{1 + \gamma_{mm}}$ , is relatively simple to satisfy, since in practice it is possible to achieve  $\gamma_{bb} = 1$  [5], and the condition is satisfied for downlink SINR (SINR at the MS)  $\geq 0 \text{dB}$ .

## 6. TWO UNIDIRECTIONAL LINKS

Much of the analysis for use case (i) (Section 5) extends to use case (ii) (Fig. 1(b)), due to the similarity between the sum rate as a function of transmission powers for these two use cases (see Eqs. (2) and (3)). However, there are also important differences. First, the interfering signal at MS 2 in use case (ii), unlike the self-interfering signal at the MS in the bidirectional link case, is not known at the receiver, and therefore cannot be cancelled (unless an additional channel is used, which we do not consider). Second, in use case (ii), the channel gains between stations cannot take arbitrary values. This is because the channel gains typically conform to a path loss model of propagation, where the SNR depends on distances between stations, which in turn need to satisfy the triangle inequality. The following two Lemmas are similar to Lemmas 5.1 and 5.2. We state them without proofs.

**LEMMA 6.1.** *If there exists an FD sum rate that is higher than the maximum TDD rate, then the FD sum rate is maximized at  $P_{m_1} = \bar{P}_m$  for MS 1, and  $P_b = \bar{P}_b$  for the BS.*

**LEMMA 6.2.** *FD extends the TDD capacity region by  $p \cdot 100\%$  if and only if:*

$$\frac{\log \left( 1 + \frac{\gamma_{bm_2}^{\max}}{1 + \gamma_{m_1 m_2}^{\max}} \right)}{\log(1 + \gamma_{bm_2}^{\max})} + \frac{\log \left( 1 + \frac{\gamma_{m_1 b}^{\max}}{1 + \gamma_{bb}^{\max}} \right)}{\log(1 + \gamma_{m_1 b}^{\max})} = 1 + p. \quad (9)$$

In a path loss model of propagation, wireless channel gain between two stations is a function of the distance between the stations:  $h_{uv} = \left( \frac{L}{d_{uv}} \right)^\eta$ , where  $u, v \in \{b, m_1, m_2\}$ ,  $u \neq v$ ,  $\eta$  is the path loss exponent, and  $L$  is a constant. Therefore, since distances  $d_{m_1 b}$ ,  $d_{bm_2}$ , and  $d_{m_1 m_2}$  need to satisfy the triangle inequality, SNRs  $\gamma_{m_1 b}$ ,  $\gamma_{bm_2}$  and INR  $\gamma_{m_1 m_2}$  cannot take arbitrary values.

To evaluate rate gains in use case (ii), we consider path loss exponents from the set  $\{2, 3, 4\}$ , since typical range for the path loss exponent is between 2 and 4 [24]. We assume fixed maximum powers at the BS and the MS 1, equal noise levels  $N$  at the BS and the MS 2, and we vary SNRs and the INR as the function of distance, as follows:

$$\begin{aligned} \gamma_{m_1 b} &= \frac{h_{m_1 b} \bar{P}_{m_1}}{N} = \frac{h_{m_1 b}}{h_{m_1 b}^{\max}} \cdot \gamma_{m_1 b}^{\max} = \left( \frac{d_{m_1 b}}{d_{m_1 b}^{\min}} \right)^\eta \gamma_{m_1 b}^{\max}, \\ \gamma_{m_1 m_2} &= \frac{h_{m_1 m_2} \bar{P}_{m_1}}{N} = \frac{h_{m_2 m_2}}{h_{m_1 m_2}^{\max}} \cdot \gamma_{m_1 m_2}^{\max} = \left( \frac{d_{m_1 m_2}}{d_{m_1 m_2}^{\min}} \right)^\eta \gamma_{m_1 m_2}^{\max}, \\ \gamma_{bm_2} &= \frac{h_{bm_2} \bar{P}_b}{N} = \frac{h_{bm_2}}{h_{bm_2}^{\max}} \cdot \gamma_{bm_2}^{\max} = \left( \frac{d_{bm_2}}{d_{bm_2}^{\min}} \right)^\eta \gamma_{bm_2}^{\max}, \end{aligned}$$

where  $d_{uv, \min}$  is a reference distance at which  $\gamma_{uv} = \gamma_{uv}^{\max}$  for  $u, v \in \{b, m_1, m_2\}$ ,  $x \neq y$ .

For the purpose of comparison, we will assume that  $d_{bm_2}^{\min} = d_{m_1 m_2}^{\min} = d^{\min}$ , which would correspond to  $\bar{P}_b = \bar{P}_m$ , and normalize all distances to  $d^{\min}$  (or, equivalently, take  $d^{\min} = 1$ ). Note that by assuming  $\bar{P}_b = \bar{P}_m$  we estimate an upper bound on the capacity region extension, since in practice  $\bar{P}_m \leq \bar{P}_b$ , and from Lemma 6.1 if FD rate can be greater than the maximum TDD rate, it is maximized when the transmission powers are maximized.

Capacity region extension as a function of SNRs is shown in Fig. 8, for different values of the path loss exponent and

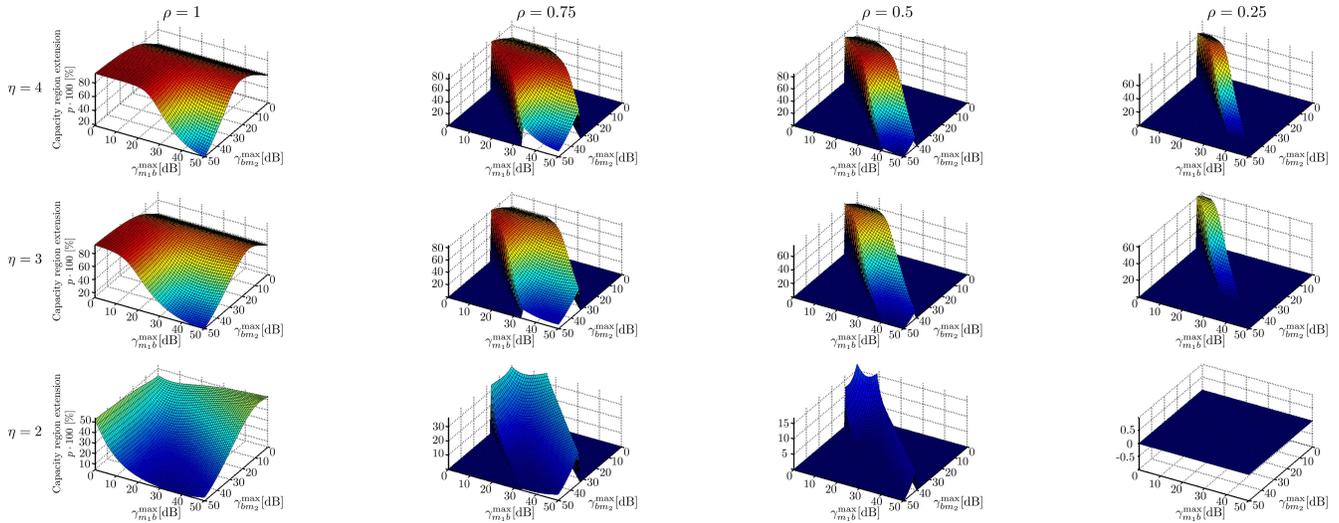


Figure 8: TDD capacity region extension due to FD as a function of SNRs, where SNRs change due to path loss with exponent  $\eta$ , and distance between MS 1 and MS 2 is  $d_{m_1 m_2} = \rho(d_{m_1 b} + d_{b m_2})$ . Transmission powers are set to maximum. In SNR regions where the triangle inequality of the distances is not satisfied,  $p$  is set to 0.

$d_{m_1 m_2} = \rho(d_{m_1 b} + d_{b m_2})$ , for  $\rho \in \{0.25, 0.5, 0.75, 1\}$ . For all combinations of SNRs at which the triangle inequality is not satisfied, we set the capacity region extension  $p$  to 0.

Fig. 8 suggests that to achieve over 50% capacity region extension, the environment needs to be sufficiently lossy, e.g., with the path loss exponent  $\eta > 2$ . Moreover, to achieve high capacity region extension, the SNRs at the BS and at the MS 2 need to be low enough, meaning that the corresponding distances  $d_{m_1 b}$  and  $d_{b m_2}$  need to be large, since the differences in the SNR shown in all the graphs are due to different distances (and consequently different path loss).

## 7. OFDM BIDIRECTIONAL LINKS

In this section, we focus on the rate maximization for use case (iii) (Fig. 1(c)). Recall that in this use case FD transceiver at the MS has a frequency-selective SIC profile (Fig. 4(d)). Requiring two technical conditions (Conditions 7.2 and 7.4), we derive a nearly optimal algorithm for the sum rate maximization. While the derived algorithm is nearly optimal and has polynomial time complexity, its running time is high because it requires invoking a large number of gradient ascent methods. We therefore also consider a high SINR approximation of the sum rate. Interestingly, our numerical results (Section 7.3) suggest that for FD capacity region extension of over 50%, high SINR approximation provides a solution that is very close to the optimal one.

### 7.1 Analysis of Sum Rate

#### 7.1.1 Dependence on Channel Power Levels

The analysis of the sum rate in terms of transmission power levels extends from the single-channel case provided in Section 5. In particular:

OBSERVATION 7.1. If

$$\frac{g_m(k-c)^2}{N_m} \leq \frac{h_{mb,k}}{N_b + g_b P_{b,k}} \text{ and } \frac{g_b}{N_b} \leq \frac{h_{bm,k}}{N_m + g_m P_{m,k}(k-c)^2} \quad (10)$$

holds, then the sum rate is concave in both  $P_{m,k}$  and  $P_{b,k}$ .

This result is simple to show, since  $(k-c)^2$  term is independent of the transmission powers, and  $P_{b,k}$  and  $P_{m,k}$  only appear in one summation term ( $r_k$ ), and therefore we get the same form of partial derivatives in  $P_{b,k}$  and  $P_{m,k}$  as in the case of a single channel (proof of Lemma 5.1). Similar to the case of a single channel, if condition (10) is not satisfied, then the achievable rate improvement is low.

The first inequality in (10) guarantees concavity in  $P_{m,k}$ , while the second one guarantees concavity in  $P_{b,k}$ . The condition (10) cannot be satisfied for any  $P_{b,k} \geq 0$ ,  $P_{m,k} \geq 0$  (e.g., the first inequality cannot be satisfied if  $\frac{g_m(k-c)^2}{N_m} > \frac{h_{mb,k}}{N_m}$ ). However, since the role of condition (10) is to guarantee concavity in the powers, we can replace this condition by either  $P_{m,k} = 0$  or  $P_{b,k} = 0$ , which implies rate concavity in  $P_{m,k}, P_{b,k}$ . Specifically, the required condition is:

CONDITION 7.2. (a)  $\frac{g_m(k-c)^2}{N_m} \leq \frac{h_{mb,k}}{N_b + g_b P_{b,k}}$  if  $\frac{g_m(k-c)^2}{N_m} < \frac{h_{mb,k}}{N_b}$ , otherwise  $P_{m,k} = 0$ , and (b)  $\frac{g_b}{N_b} \leq \frac{h_{bm,k}}{N_m + g_m P_{m,k}(k-c)^2}$  if  $\frac{g_b}{N_b} < \frac{h_{bm,k}}{N_m}$ ; otherwise  $P_{b,k} = 0$  if  $P_{m,k}$  was not set to 0 by (a).

#### 7.1.2 Dependence on Maximum SIC Frequency

The following lemma shows that choosing optimal  $c$  for a given power allocation  $\{P_{b,k}, P_{m,k}\}$  is hard in general, since sum rate  $r$  as a function of  $c$  is neither convex nor concave, and can have  $\Omega(K)$  local maxima. Proofs of the following two lemmas are provided in the Appendix.

LEMMA 7.3. The sum rate  $r$  is neither convex nor concave in  $c$ . All (local) maxima of  $r(c)$  lie in the interval  $(1, K)$ . In general, the number of local maxima is  $\Omega(K)$ .

Even though  $r(c)$  can have multiple maxima in  $c$ , if we restrict the analysis to practically relevant values of  $\gamma_{mb,k}$  and  $\gamma_{mm,k}$ , the selection of  $c$ , together with the power allocation, can be made tractable under the following condition:

CONDITION 7.4.  $\frac{g_m P_{m,k}}{N_m} \leq \gamma_{mb,k} \forall k \in \{1, \dots, K\}$ .

Note that this condition is implied by Condition 7.2 for  $|k-c| \geq 1$ , and that there can be at most 2 channels with  $|k-c| < 1$ . For  $|k-c| < 1$ , this condition limits SI on channel  $k$ . The following lemma bounds the first partial derivative of  $r$  with respect to  $c$ . This bound will prove useful in maximizing  $r$  as a function of  $c$  and  $\{P_{b,k}, P_{m,k}\}$  (Section 7.2.1).

LEMMA 7.5. *If Condition 7.4 holds then:*

$$\left| \frac{\partial r}{\partial c} \right| \leq \frac{2}{\ln 2} (\ln(K) + 1 + 2\sqrt{3}) \quad \forall c \in (1, K).$$

## 7.2 Optimal Parameter Selection

In this section we provide a nearly optimal algorithm for determining  $c, \{P_{b,k}, P_{m,k}\}$  that maximize the sum rate under conditions 7.2 and 7.4. As argued before, these two conditions impose mild restrictions on the maximum rate.

While the algorithm we develop for the general SINR regime runs in polynomial time with an arbitrary (constant) accuracy, its running time is high, which makes it unsuitable for real-time applications. However, this algorithm allows us to estimate the achievable FD rate gains.

To characterize properties of the power allocation and parameter  $c$  selection in the high SINR regime, we demonstrate that under high SINR approximation of the rate it is always optimal to place maximum SIC in the middle of the frequency band, and in that case the power allocation both at the BS and at the MS is fixed and can be determined efficiently (Section 7.2.2).

### 7.2.1 General SINR Regime

The algorithm for maximizing the sum rate in general SINR regime is provided below.

---

#### Algorithm 1 MAXIMUMRATE( $\epsilon$ )

---

Input:  $K, \bar{P}_b, \bar{P}_m, g_b, g_m, N_m, N_b$   
1:  $c_{\min} = 1, c_{\max} = K, \Delta c = \frac{\epsilon}{\frac{2}{\ln 2} (\ln(K) + 1 + 2\sqrt{3})}$   
2:  $c^{\max} = r^{\max} = 0, \{P_{b,k}^{\max}\} = \{P_{m,k}^{\max}\} = \{0\}$   
3: **for**  $c = c_{\min}, c < c_{\max}, c = c + \Delta c$  **do**  
4:   Solve via gradient ascent:  
    $\max r = \sum_{k=1}^K r_k$ , where  $r_k$  is given by (4)  
   **s.t.** Conditions 7.2 and 7.4 hold  
    $\sum_{k=1}^K P_{m,k} \leq \bar{P}_m, \sum_{k=1}^K P_{b,k} \leq \bar{P}_b$   
    $P_{b,k} \geq 0, P_{m,k} \geq 0 \quad \forall k \in \{1, \dots, K\}$ .  
5:   **if**  $r > r^{\max}$  **then**  
6:      $r^{\max} = r, c^{\max} = c,$   
7:      $\{P_{b,k}^{\max}\} = \{P_{b,k}\}, \{P_{m,k}^{\max}\} = \{P_{m,k}\}$   
8: **return**  $c^{\max}, \{P_{b,k}^{\max}\}, \{P_{m,k}^{\max}\}, r^{\max}$ .

---

We claim the following:

LEMMA 7.6. *Under conditions 7.2 and 7.4, MAXIMUMRATE determines  $c$  and the power allocation  $\{P_{b,k}, P_{m,k}\}$  that maximize sum rate up to an absolute error  $\epsilon$ , for any  $\epsilon > 0$ . The running time of MAXIMUMRATE is polynomial in  $\frac{1}{\epsilon}$  and  $K$ .*

Note that if we only applied gradient ascent to maximize the sum rate without requiring condition 7.2, we would not necessarily obtain an optimal power allocation, since without condition 7.2 the sum rate is not necessarily concave in the power variables. Moreover, since the sum rate is highly nonlinear in the parameter  $c$  (Lemma 7.3), we cannot use  $c$

as a variable in the gradient ascent. Nevertheless, as a result of Lemma 7.5 that bounds the first derivative of  $r$  with respect to  $c$  when condition 7.4 is applied, by examining a sufficient number of choices for  $c$  from the interval  $(1, K)$  and determining an optimal power allocation for each such  $c$ , we obtain a nearly-optimal solution.

PROOF OF LEMMA 7.6. First, consider the optimization problem in step 4 of the algorithm. Since Condition 7.2 is required by the constraints, the objective  $r$  is concave in  $P_{b,k}, P_{m,k}$ . The feasible region of the problem from Step 4 is determined by linear inequalities and 7.2 and 7.4.

Condition 7.2 is either an inequality or an equality for each  $P_{m,k}, P_{b,k}$  that (possibly rearranging the terms) is linear in  $P_{m,k}, P_{b,k}$ . Condition 7.4 is a linear inequality in  $P_{m,k}$ . Therefore, the feasible region in the problem of Step 4 is a polyhedron and therefore convex. It follows immediately that this problem can be solved optimally via a gradient ascent method in time polynomial in  $K$ .

The algorithm finds an optimal power allocation for each  $c$  from the set of  $\frac{(K-1)(\frac{2}{\ln 2} (\ln(K) + 1 + 2\sqrt{3}))}{\epsilon} - 2$  equally spaced points from the interval  $(1, K)$ , and chooses  $c$  and power allocation that provide maximum sum rate  $r$ .

What remains to prove is that by choosing any alternative  $c \neq c^{\max}$  and accompanying optimal power allocation the sum rate cannot be improved by more than an additive  $\epsilon$ .

Recall from Lemma 7.3 that optimal  $c$  must lie in  $(1, K)$ . Suppose that there exist  $c^*, \{P_{b,k}^*, P_{m,k}^*\}$  such that  $c^* \in (1, K), c^* \neq c^{\max}$  and  $r(c^*, \{P_{b,k}^*, P_{m,k}^*\}) > r^{\max} + \epsilon$ .

From the choice of points  $c$  in the algorithm, there must exist at least one point  $c^a$  that the algorithm considers such that  $|c^a - c^*| < \Delta c = \frac{\epsilon}{\frac{2}{\ln 2} (\ln(K) + 1 + 2\sqrt{3})}$ . From Lemma 7.5,

$$r(c^*, \{P_{b,k}^*, P_{m,k}^*\}) - r(c^a, \{P_{b,k}^*, P_{m,k}^*\}) < \frac{\epsilon}{\frac{2}{\ln 2} (\ln(K) + 1 + 2\sqrt{3})} \cdot \left( \frac{2}{\ln 2} (\ln(K) + 1 + 2\sqrt{3}) \right) = \epsilon,$$

since in any finite interval  $I$  any continuous and differentiable function  $f(x)$  cannot change by more than the length of the interval  $I$  times the maximum value of its first derivative  $f'(x)$  (a simple corollary of the Mean-Value Theorem).

Since the algorithm finds an optimal power allocation for each  $c$ , we have that  $r(c^a, \{P_{b,k}^*, P_{m,k}^*\}) \leq r(c^a, \{P_{b,k}^a, P_{m,k}^a\}) \leq r^{\max}$ . Therefore:  $r(c^*, \{P_{b,k}^*, P_{m,k}^*\}) - r^{\max} < \epsilon$ , which is a contradiction.  $\square$

### 7.2.2 High SINR Regime

A high SINR approximation of the sum rate is:

$$r \approx \sum_{k=1}^K \left( \log \left( \frac{\gamma_{mb,k}}{1 + \gamma_{bb,k}} \right) + \log \left( \frac{\gamma_{bm,k}}{1 + \gamma_{mm,k}} \right) \right) \quad (11)$$

While in the high SINR the dependence of sum rate on each power  $P_{b,k}, P_{m,k}$  for  $k \in \{1, \dots, K\}$  becomes concave (regardless of whether condition 7.2 holds or not), the dependence on the parameter  $c$  remains neither convex nor concave as long as we consider a general power allocation. Therefore, one cannot derive a closed form expression for  $c$  in terms of an arbitrary power allocation. However, as we show in Lemma 7.9, when power allocation and the choice of parameter  $c$  are considered jointly, it is always optimal to place  $c$  in the middle of the interval  $(1, K)$ :  $c = \frac{K+1}{2}$ .

The following proposition and lemma characterize the optimal power allocation for a given  $c$ .

PROPOSITION 7.7. *Under high-SINR approximation and any power allocation  $\{P_{m,k}\}$  at the MS and any choice of  $c$ , it is always optimal to allocate BS powers as  $P_{b,k} = \frac{\bar{P}_b}{K}$ .*

PROOF. Let  $P_b$  denote the total irradiated power by the BS. Write powers on individual subchannels as  $P_{b,k} = \beta_k P_b$ , where  $\beta_k \geq 0 \forall k \in \{1, \dots, K\}$  and  $\sum_{k=1}^K \beta_k = 1$ . Then the sum rate can be written as:

$$r = \sum_{k=1}^K \left( \log \left( \frac{h_{mb,k} P_{m,k}}{N_b + g_b \beta_k P_b} \right) + \log \left( \frac{h_{bm,k} \beta_k P_b}{N_m + g_m (k-c)^2 P_{m,k}} \right) \right)$$

First, observe that

$$\begin{aligned} \frac{\partial r}{\partial P_b} &= \sum_{k=1}^K \mathbb{1}_{\{\beta_k > 0\}} \left( \frac{1}{P_b} - \frac{g_b \beta_k}{N_b + g_b \beta_k P_b} \right) \\ &= \sum_{k=1}^K \mathbb{1}_{\{\beta_k > 0\}} \left( \frac{1}{P_b} - \frac{1}{\frac{N_b}{g_b \beta_k} + P_b} \right), \end{aligned}$$

where  $\mathbb{1}_{\{\cdot\}}$  is an indicator function. Since  $\beta_k \geq 0 \forall k \in \{1, \dots, K\}$  and  $\sum_{k=1}^K \beta_k = 1$ , it follows that there exists at least one strictly positive  $\beta_k$ . For each such  $\beta_k$ ,  $\mathbb{1}_{\{\beta_k > 0\}} \left( \frac{1}{P_b} - \frac{1}{\frac{N_b}{g_b \beta_k} + P_b} \right) > 0$ , since  $P_b < \frac{N_b}{g_b \beta_k} + P_b$ . Therefore,  $\frac{\partial r}{\partial P_b} > 0$ , which implies that it is optimal to choose  $P_b = \bar{P}_b$ .

Taking the first and the second partial derivative of  $r$  with respect to each  $\beta_k$ , it is simple to show that  $r$  has the same dependence on each  $\beta_k$ , and, moreover, is strictly concave in each  $\beta_k$ , as  $\frac{\partial^2 r}{\partial \beta_k^2} = -\frac{1}{\beta_k^2} + \frac{1}{\left(\frac{N_b}{g_b \beta_k} + \beta_k\right)^2} < 0$ , where the inequality follows from  $\beta_k < \frac{N_b}{g_b P_b} + \beta_k$ . Therefore,  $r$  is maximized for  $\beta_k = \frac{1}{K}$ .  $\square$

LEMMA 7.8. *Under high-SINR approximation and for a given, fixed,  $c$  the optimal power allocation at the MS satisfies  $P_{m,k} = \alpha_k \cdot \bar{P}_m$ , where  $\alpha_k \geq 0$ ,  $\sum \alpha_k = 1$ , and for  $k \neq K$ :*

- (i)  $\alpha_k = \left( \frac{1}{\alpha_K} - \frac{1}{N_m/R_K + \alpha_K} \right)^{-1}$  if  $k = c$ ,
  - (ii)  $\alpha_k = \frac{-N_m + \sqrt{N_m^2 + 4\alpha_K(N_m + R_K\alpha_K)R_k}}{2R_k}$  if  $k \neq c$ ,
- where  $R_k = g_m(k-c)^2 P_m$  for  $k \in \{1, \dots, K\}$ .

PROOF. Let  $P_{m,k} = \alpha_k P_m$ , where  $\alpha_k > 0 \forall k$  and  $\sum_{k=1}^K \alpha_k = 1$ . The sum rate can then be written as:

$$r = \sum_{k=1}^K \left( \log \left( \frac{h_{mb,k} \alpha_k P_m}{N_b + g_b P_{b,k}} \right) + \log \left( \frac{h_{bm,k} P_{b,k}}{N_m + g_m (k-c)^2 \alpha_k P_m} \right) \right).$$

Proving that at the optimal solution that maximizes  $r$  we necessarily have  $P_m = \bar{P}_m$  is analogous to the proof given in Proposition 7.7 for  $P_b = \bar{P}_b$ , and it is therefore omitted.

As  $\sum_{k=1}^K \alpha_k = 1$ , only  $K-1$   $\alpha_k$ 's can be chosen independently, while the value of the remaining one is implied by their sum being equal to 1. Choose  $\alpha_K = 1 - \sum_{k=1}^{K-1} \alpha_k$ , and observe that that  $K-c \neq 0$  (and therefore  $R_K \neq 0$ ) is always true since, similar as in proof of Lemma 7.3, at the optimum it must be  $c \in (1, K)$ .

If  $R_k = g_m(k-c)^2 P_m = 0$ , then the first and the second derivative of  $r$  with respect to  $\alpha_k$  are given as:

$$\begin{aligned} \frac{\partial r}{\partial \alpha_k} &= \frac{1}{\alpha_k} + \frac{1}{\alpha_K} \frac{\partial \alpha_K}{\partial \alpha_k} - \frac{R_K}{N_m + R_K \alpha_K} \frac{\partial \alpha_K}{\partial \alpha_k} \\ &= \frac{1}{\alpha_k} - \frac{1}{\alpha_K} + \frac{R_K}{N_m + R_K \alpha_K}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 r}{\partial \alpha_k^2} &= -\frac{1}{\alpha_k^2} - \left( \frac{1}{\alpha_K^2} - \frac{R_K^2}{(N_m + R_K \alpha_K)^2} \right) \\ &= -\frac{1}{\alpha_k^2} - \left( \frac{1}{\alpha_K^2} - \frac{1}{(N_m/R_K + \alpha_K)^2} \right) < 0. \end{aligned}$$

It follows that  $r$  is concave in  $\alpha_k$  and maximized for

$$\alpha_k = \left( \frac{1}{\alpha_K} - \frac{1}{N_m/R_K + \alpha_K} \right)^{-1}, \quad (12)$$

where  $R_K = g_m(K-c)^2 P_m$ .

If  $R_k \neq 0$ , then the first and the second derivative are:

$$\frac{\partial r}{\partial \alpha_k} = \frac{1}{\alpha_k} - \frac{R_k}{N_i + R_k \alpha_k} - \frac{1}{\alpha_K} + \frac{R_K}{N_m + R_K \alpha_K},$$

$$\begin{aligned} \frac{\partial^2 r}{\partial \alpha_k^2} &= -\frac{1}{\alpha_k^2} + \frac{R_k^2}{(N_m + R_k \alpha_k)^2} - \frac{1}{\alpha_K^2} + \frac{R_K^2}{(N_m + R_K \alpha_K)^2} \\ &= -(\alpha_k^{-2} - (N_m/R_k + \alpha_k)^{-2}) \\ &\quad - (\alpha_K^{-2} - (N_m/R_K + \alpha_K)^{-2}) < 0. \end{aligned}$$

It follows that  $r$  is concave in  $\alpha_k$  and maximized for:

$$\frac{\partial r}{\partial \alpha_k} = \frac{1}{\alpha_k} - \frac{R_k}{N_i + R_k \alpha_k} - \frac{1}{\alpha_K} + \frac{R_K}{N_m + R_K \alpha_K} = 0. \quad (13)$$

After simplifying (13), we get:

$$\alpha_k(N_m + R_k \alpha_k) = \alpha_K(N_m + R_K \alpha_K). \quad (14)$$

Solving the quadratic equation (14) for  $\alpha_k$  and using that  $\alpha_k > 0$ , it follows that  $r$  is maximized when  $\alpha_k$  satisfies

$$\alpha_k = \frac{-N_m + \sqrt{N_m^2 + 4\alpha_K(N_m + R_K \alpha_K)R_k}}{2R_k}, \quad (15)$$

where  $R_k = g_m(k-c)^2 P_m$ .  $\square$

It is relatively simple to show (using similar approach as in the proof of Lemma 7.3) that under general power allocation  $r$  can have up to  $K$  local maxima with respect to  $c$ . However, if  $c$  is considered jointly with the optimal power allocation corresponding to  $c$  (Proposition 7.7 and Lemma 7.8) it is always optimal to place  $c$  in the middle of the interval  $(1, K)$ , as the following lemma states.

Even though this result may seem intuitive because the optimal power allocation is always symmetric around  $c$  (Proposition 7.7 and Lemma 7.8), the proof does not follow directly from this property and requires many technical details. For this reason, the proof is deferred to the Appendix.

LEMMA 7.9. *If  $(c, \{P_{b,k}, P_{m,k}\})$  maximizes the sum rate under high SINR approximation, then  $c = \frac{K+1}{2}$ .*

A simple corollary of Lemma 7.9 is that:

COROLLARY 7.10. *If  $(c^*, \{P_{m,k}^{\max}, P_{b,k}^{\max}\})$  maximizes  $r$  under high SINR approximation, then the power allocation  $\{P_{m,k}^{\max}\}$  is symmetric around  $\frac{K+1}{2}$  and decreasing in  $|k-c|$ .*

PROOF. The first part follows directly from  $c^{\max} = \frac{K+1}{2}$ . The second part is proved in Lemma 7.9, claim (K2).  $\square$

LEMMA 7.11. *A solution  $(c^{\max}, \{P_{m,k}^{\max}, P_{b,k}^{\max}\})$  that maximizes  $r$  under high SINR approximation up to an absolute error  $\epsilon$  can be computed in  $O(K \log(\frac{1}{\epsilon}))$  time.*

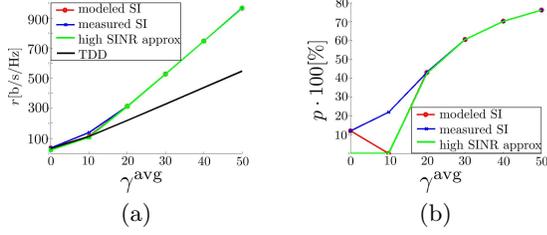


Figure 9: Evaluated (a) sum rate and (b) capacity region extension as a function of  $\gamma^{\text{avg}} = \frac{h_{m,b,k} \frac{1}{K}}{N_m} = \frac{h_{m,b,k} \frac{1}{K}}{N_b}$ . For  $\gamma^{\text{avg}} \geq 20$ dB, sum rate and capacity region extension for MAXIMUMRATE with measured SI, MAXIMUMRATE with modeled SI, and high SINR approximation match closely.

PROOF. From Proposition 7.7, at the optimum  $P_{b,k}^{\text{max}} = \frac{P_{b,k}^{\text{max}}}{K}$ ,  $\forall k \in \{1, \dots, K\}$ . This can be computed in constant time, and requires  $\Theta(K)$  time to assign the values to all the  $P_{b,k}$ 's. From Lemma 7.9,  $c^{\text{max}} = \frac{K+1}{2}$ .

From Lemma 7.8,  $P_{m,k}^{\text{max}} = \alpha_k \cdot \bar{P}_m$ , where  $\{\alpha_k\}$  are positive coefficients given by (12), (15) and  $\sum_{k=1}^K \alpha_k = 1$ . Recall that all the  $\alpha_k$ 's are given in terms of  $\alpha_K$ , so we can determine the allocation  $\{\alpha_k\}$  by performing a binary search for  $\alpha_K$  until  $\sum_{k=1}^K \alpha_k \in [1 - \epsilon', 1]$ . Corollary 7.10 implies that  $\alpha_K \leq \frac{1}{K}$ , so it is sufficient to perform the binary search for  $\alpha_K$  only within the interval  $[0, \frac{1}{K}]$ . Such binary search requires  $O(\log(\frac{1}{K\epsilon'}))$  iterations, with each iterations requiring  $O(K)$  time to compute  $\{\alpha_k\}$  and evaluate  $\sum_{k=1}^K \alpha_k$ , for the total time  $O(K \log(\frac{1}{K\epsilon'}))$ .

The last part of the proof is determining an appropriate  $\epsilon'$  so that  $r(c^{\text{max}}, \{P_{m,k}^{\text{max}}, P_{b,k}^{\text{max}}\}) \geq \max r - \epsilon$ , where the maximum is taken over all feasible points  $(c, \{P_{m,k}, P_{b,k}\})$ . Notice that we are only deviating from the optimal solution in that  $\sum_{k=1}^K P_{m,k}^{\text{max}} = \bar{P}_m \cdot \sum_{k=1}^K \alpha_k \in [\bar{P}_m(1 - \epsilon'), \bar{P}_m]$  instead of  $\sum_{k=1}^K P_{m,k}^{\text{max}} = \bar{P}_m$ . Therefore,  $(c^{\text{max}}, \{P_{m,k}^{\text{max}}, P_{b,k}^{\text{max}}\})$  is the optimal solution to the problem that is equivalent to the original problem, with maximum total power at the MS equal to  $\bar{P}_m \cdot \sum_{k=1}^K \alpha_k$ . Observe that:

$$\frac{\partial r}{\partial P_m} = \sum_{k=1}^K \left( \frac{1}{P_m} - \frac{1}{\frac{N_m}{g_m(k-c)^2} + P_i} \cdot \mathbb{1}_{\{k \neq c\}} \right) \leq \frac{K}{P_m}.$$

As  $\frac{\partial r}{\partial P_m}(P_m) \leq \frac{K}{P_m(1-\epsilon')}$  for  $P_m \in [\bar{P}_m(1-\epsilon'), \bar{P}_m]$ , it follows that:  $\max r - r(c^{\text{max}}, \{P_{m,k}^{\text{max}}, P_{b,k}^{\text{max}}\}) \leq \frac{K}{P_m(1-\epsilon')} \cdot \bar{P}_m \epsilon'$ . Setting:  $\frac{K}{P_m(1-\epsilon')} \cdot \bar{P}_m \epsilon' = \epsilon \Leftrightarrow \epsilon' = \frac{\epsilon}{K+\epsilon}$ , we yield the total running time of:  $O(K \log(\frac{K+\epsilon}{K\epsilon})) = O(K \log(\frac{1}{\epsilon}))$ .  $\square$

### 7.3 Measurement-based Numerical Evaluation

This section presents numerical evaluations for use case (iii). Numerical evaluations for use cases (i) and (ii) were already provided in Sections 5 and 6, respectively.

We focus on the impact of a frequency-selective SIC profile in a small form factor hardware at the MS (Fig. 4(d)), and evaluate rate gains based on measurements and the model presented in Sections 3 and 4.1, respectively.

To determine the position  $c^{\text{max}}$  of maximum SIC and power allocation  $\{P_{m,k}^{\text{max}}, P_{b,k}^{\text{max}}\}$  that maximize sum rate, we run an implementation of the MAXIMUMRATE algorithm separately for measured and modeled SIC profile of the MS FD

transceiver. Additionally, we determine  $c^{\text{max}}, \{P_{m,k}^{\text{max}}, P_{b,k}^{\text{max}}\}$  for the high SINR approximation of the sum rate using the analytical results from Section 7.2.2.

Since the measurements were performed only for the analog part of the FD transceiver, we assume additional 50dB of cancellation from the digital domain.<sup>5</sup> Similar to [5], we assume that when either station transmits at maximum total power that is equally allocated across channels (so that  $P_{m,k} = \frac{\bar{P}_m}{K}, P_{b,k} = \frac{\bar{P}_b}{K}$ ), the noise on each channel is 110dB below the transmitted power level.

We consider total bandwidth of 20MHz in the range 2.13–2.15GHz, and take the distance between the measurement points as the OFDM channel width ( $\approx 600$ kHz), so that there are  $K = 33$  channels in the considered band. For the SIC at the BS, we take  $g_b = 1$  [5].

We scale all the power variables so that  $\bar{P}_m = \bar{P}_b = 1$ , and normalize SNR to the scaled values of power levels on channels. We consider flat frequency fading, and perform numerical evaluations for  $\frac{h_{m,b,k}}{N_b} = \frac{h_{b,m,k}}{N_m} \equiv \gamma^{\text{avg}} \cdot K \forall k$ . We perform numerical evaluations for  $\gamma^{\text{avg}} \in \{0, 10, 20, 30, 40, 50\}$  [dB].

We run MAXIMUMRATE for  $\Delta c = 0.01$ , which translates into an absolute error  $\epsilon \approx 0.2$  for  $r$ . We evaluate the sum rate and the capacity region extension using the measurement data for the remaining SI and  $c^{\text{max}}, \{P_{m,k}^{\text{max}}, P_{b,k}^{\text{max}}\}$  returned by the algorithm. We assume that the SIC profile does not change as  $f_c$  (and correspondingly  $c$ ) is varied. To run the algorithm for  $c$  positioned at any point between two neighboring channels, we interpolate the measurement data.

Since for the transmitted power of  $1/K$  and  $c$  placed in the middle of the frequency band XINR at the first and the last channel is about 35 ( $\approx 15$ dB), our numerical results suggest, as expected (see e.g., Fig. 5(b) and Fig. 6(b)), that to achieve high rate gain,  $\gamma^{\text{avg}}$  needs to be sufficiently high.

In particular, for  $\gamma^{\text{avg}}$  up to 10dB the algorithm selects most of the channels as half-duplex, where either  $P_{m,k}^{\text{max}} = 0$  or  $P_{b,k}^{\text{max}} = 0$  (Fig. 10(a), (b)), and therefore capacity region extension is up to about 20% and additive increase in sum rate is small (Fig. 10(a), (b)).

Already at  $\gamma^{\text{avg}} = 20$ dB, all except for three channels are operating in FD (Fig. 10(a), (b)), and high SINR approximation solution provides the rate and capacity region extension that closely match those computed by the MAXIMUMRATE algorithm run both for measured and modeled SIC profiles (Fig. 9(a), (b)). For  $\gamma^{\text{avg}} \geq 30$ dB, power allocation, sum rate, and capacity region extension for high SINR approximation and MAXIMUMRATE algorithm are almost indistinguishable, and result in capacity region extension of 60-80% (Fig. 10(d), (c) and Fig.9(a), (b)).

## 8. CONCLUSION AND FUTURE WORK

In this paper we considered three basic use cases of FD, including single- and multi-channel scenarios. In order to analyze the multi-channel scenario, we developed a new model that is grounded in realistic circuit implementations of small form factor devices. We characterized the rate gains in different scenarios and solved power allocation and frequency selection problems either analytically or algorithmically. Our numerical results demonstrate the gains from FD in scenarios and for circuits that have not been studied before.

<sup>5</sup>Note that Fig. 4(d) only shows isolation that results from the SIC in the analog domain.

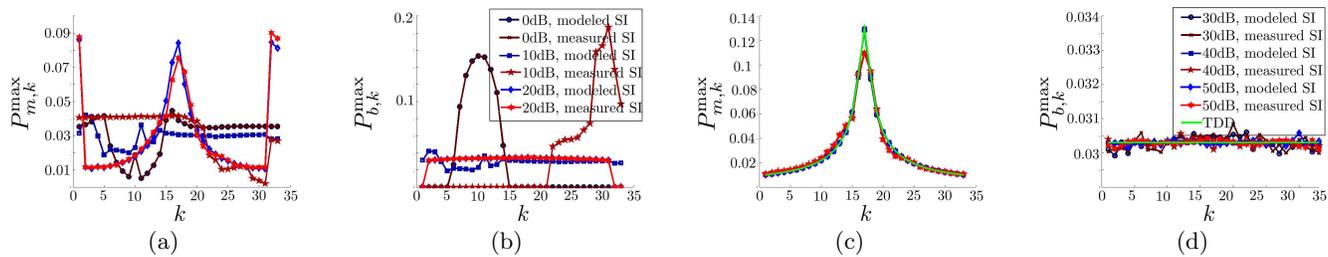


Figure 10: Power allocation over channels for SNR (a), (b) 0dB–20dB and (c), (d) 30dB–50dB. The graphs show the results obtained by running MAXIMUMRATE with measured SI and modeled SI at the input, and by using results from Section 7.2.2 for the high SINR approximation. For  $\text{SNR} \geq 30\text{dB}$  all three power allocations match closely.

This is one of the first steps towards understanding the benefits and the complexities associated with FD. Hence, there are still many open problems to consider. In particular, SIC that has different impacts on different channels calls for the design of algorithms for OFDM networks with multiple access and MSs modeled as small form-factor devices. Moreover, we plan to develop scheduling algorithms that support the co-existence of half- and full-duplex users. While significant attention has been given to scheduling and resource allocation in half duplex OFDM networks [6, 16], as demonstrated in this paper, the special characteristics of FD pose new challenges that have not been addressed.

## 9. REFERENCES

- [1] Skyfr-000709 miniature 2110-2170MHz single junction robust lead circulator data sheet.
- [2] E. Ahmed, A. M. Eltawil, and A. Sabharwal. Rate gain region and design tradeoffs for full-duplex wireless communications. *IEEE Trans. Wireless Commun.*, 12(7):3556–3565, 2013.
- [3] E. Aryafar, M. A. Khojastepour, K. Sundaresan, S. Rangarajan, and M. Chiang. MIDU: enabling MIMO full duplex. In *Proc. ACM MobiCom’12*, 2012.
- [4] J. Bai and A. Sabharwal. Distributed full-duplex via wireless side-channels: Bounds and protocols. *IEEE Trans. Wireless Commun.*, 12(8):4162–4173, 2013.
- [5] D. Bharadia, E. McMillin, and S. Katti. Full duplex radios. In *Proc. ACM SIGCOMM’13*, 2013.
- [6] S. C. Borst, M. G. Markakis, and I. Saniee. Distributed power allocation and user assignment in OFDMA cellular networks. In *IEEE Allerton’11*, 2011.
- [7] W. Cheng, X. Zhang, and H. Zhang. Optimal dynamic power control for full-duplex bidirectional-channel based wireless networks. In *Proc. IEEE INFOCOM’13*, 2013.
- [8] J. Choi, S. Hong, M. Jain, S. Katti, P. Levis, and J. Mehlman. Beyond full duplex wireless. In *Proc. IEEE Asilomar’12*, 2012.
- [9] J. Choi, M. Jain, K. Srinivasan, P. Levis, and S. Katti. Achieving single channel, full duplex wireless communication. In *Proc. ACM MobiCom’10*, 2010.
- [10] B. Debaillie, D.-J. van den Broek, C. Lavin, B. van Liempd, E. Klumperink, C. Palacios, J. Craninckx, B. Nauta, and A. Parssinen. Analog/RF solutions enabling compact full-duplex radios. *IEEE J. Sel. Areas Commun.*, 32(9):1662–1673, Sept 2014.
- [11] M. Duarte, C. Dick, and A. Sabharwal. Experiment-driven characterization of full-duplex wireless systems. *IEEE Trans. Wireless Commun.*, 11(12):4296–4307, 2012.
- [12] M. Duarte and A. Sabharwal. Full-duplex wireless communications using off-the-shelf radios: Feasibility and first results. In *Proc. IEEE Asilomar’10*, 2010.
- [13] M. Duarte, A. Sabharwal, V. Aggarwal, R. Jana, K. Ramakrishnan, C. Rice, and N. Shankaranarayanan. Design and characterization of a full-duplex multi-antenna system for WiFi networks. *IEEE Trans. Veh. Technol.*, 63(3):1160–1177, Mar. 2014.
- [14] E. Everett, A. Sahai, and A. Sabharwal. Passive self interference suppression for full-duplex infrastructure nodes. *IEEE Trans. Wireless Commun.*, 13(2):680–694, Feb. 2014.
- [15] S. Goyal, P. Liu, S. Panwar, R. DiFazio, R. Yang, J. Li, and E. Bala. Improving small cell capacity with common-carrier full duplex radios. In *Proc. IEEE ICC’14*, 2014.
- [16] J. Huang, V. G. Subramanian, R. Agrawal, and R. A. Berry. Downlink scheduling and resource allocation for OFDM systems. *IEEE Trans. Wireless Commun.*, 8(1).
- [17] M. Jain, J. Choi, T. Kim, D. Bharadia, S. Seth, K. Srinivasan, P. Levis, S. Katti, and P. Sinha. Practical, real-time, full duplex wireless. In *Proc. ACM MobiCom’11*, 2011.
- [18] A. Khandani. Two-way (true full-duplex) wireless. In *Proc. CWIT’13*, 2013.
- [19] M. Khojastepour and S. Rangarajan. Wideband digital cancellation for full-duplex communications. In *Proc. IEEE Asilomar’12*, 2012.
- [20] M. Khojastepour, K. Sundaresan, S. Rangarajan, X. Zhang, and S. Barghi. The case for antenna cancellation for scalable full-duplex wireless communications. In *Proc. ACM HotNets’11*, 2011.
- [21] M. E. Knox. Single antenna full duplex communications using a common carrier. In *Proc. IEEE WAMICON’12*, 2012.
- [22] W. Li, J. Lilleberg, and K. Rikkinen. On rate region analysis of half- and full-duplex OFDM communication links. *IEEE J. Sel. Areas Commun.*, 32(9):1688–1698, Sept. 2014.
- [23] J. McMichael and K. Kolodziej. Optimal tuning of analog self-interference cancellers for full-duplex wireless communication. In *IEEE Allerton’12*, 2012.
- [24] T. S. Rappaport. *Wireless communications: principles and practice*, chapter 4. Prentice-Hall, 2 edition, 2002.
- [25] A. Sabharwal, P. Schniter, D. Guo, D. Bliss, S. Rangarajan, and R. Wichman. In-band full-duplex wireless: Challenges and opportunities. *IEEE J. Sel. Areas Commun.*, 32(9):1637–1652, Sept. 2014.
- [26] A. Sahai, S. Diggavi, and A. Sabharwal. On uplink/downlink full-duplex networks. In *Proc. IEEE Asilomar’13*, 2013.
- [27] A. Sahai, G. Patel, C. Dick, and A. Sabharwal. Understanding the impact of phase noise on active cancellation in wireless full-duplex. In *IEEE Asilomar’12*, 2012.
- [28] A. Sahai, G. Patel, and A. Sabharwal. Pushing the limits of full-duplex: Design and real-time implementation. *arXiv preprint arXiv:1107.0607*, 2011.
- [29] N. Singh, D. Gunawardena, A. Proutiere, B. Radunovic, H. V. Balan, and P. Key. Efficient and fair MAC for wireless networks with self-interference cancellation. In *Proc. WiOpt’11*, 2011.
- [30] X. Xie and X. Zhang. Does full-duplex double the capacity of wireless networks? In *Proc. IEEE INFOCOM’14*, 2014.
- [31] R. Young. 75.9 Euler’s constant. *The Mathematical Gazette*, pages 187–190, 1991.
- [32] J. Zhou, A. Chakrabarti, P. Kinget, and H. Krishnaswamy. Low-Noise Active Cancellation of Transmitter Leakage and Transmitter Noise in Broadband Wireless Receivers for FDD/Co-Existence. *IEEE J. Solid-State Circuits*, 49(12):1–17, Dec. 2014.
- [33] J. Zhou, T.-H. Chuang, T. Dinc, and H. Krishnaswamy. A Receiver with Reconfigurable Self-Interference Cancellation Based on RF Frequency-Domain Equalization Supporting >20MHz Cancellation Bandwidth for FDD, Co-Existence and Same-channel Full Duplex Applications. In *Proc. IEEE ISSCC’15*, 2015.
- [34] J. Zhou, P. R. Kinget, and H. Krishnaswamy. A Blocker-Resilient Wideband Receiver with Low-Noise Active Two-Point Cancellation of >0dBm TX Leakage and TX Noise in RX Band for FDD/Co-Existence. In *Proc. IEEE ISSCC’14*, 2014.

## APPENDIX

PROOF OF LEMMA 7.3 . Since  $r = \sum_{k=1}^K r_k$ , we will first observe partial derivatives of  $r_k$  with respect to  $c$ .

Observe that in the expression (4) for  $r_k$  only  $\gamma_{mm,k}$  depends on  $c$ . Moreover, since  $\gamma_{mm,k} = \frac{g_m(k-c)^2 P_{m,k}}{N_m}$ , we have that  $(k-c) \frac{\partial \gamma_{mm,k}}{\partial c} = -2\gamma_{mm,k}$ .

Observe partial derivatives of  $r_k$  with respect to  $c$ :

$$\frac{\partial r_k}{\partial c} = \frac{2}{\ln 2} \cdot \frac{g_m P_{m,k}}{N_m} \cdot \gamma_{mb,k} \cdot \frac{1}{(1 + \gamma_{mb,k} + \gamma_{mm,k}(c))(1 + \gamma_{mm,k}(c))}, \quad (16)$$

$$\frac{\partial^2 r_k}{\partial c^2} = \frac{2}{\ln 2} \cdot \frac{g_m P_{m,k}}{N_m} \cdot \gamma_{mb,k} \cdot \frac{\gamma_{mm,k}(c)(2 + \gamma_{mb,k} + 3\gamma_{mm,k}(c)) - (1 + \gamma_{mb,k})}{(1 + \gamma_{mb,k} + \gamma_{mm,k}(c))^2 (1 + \gamma_{mm,k}(c))^2}. \quad (17)$$

From (16),  $\frac{\partial r_k}{\partial c}$  equals zero for  $c = k$ , it is positive for  $c < k$  and negative for  $c > k$ . Therefore,  $r_k$  has a unique maximum in  $c$ , with the maximum attained at  $k = c$ . Since this is true for every  $k \in \{1, \dots, K\}$ , it follows that for  $c \leq 1 \forall k \in \{1, \dots, K\}$ :  $\frac{\partial r_k}{\partial c} \geq 0$  (with equality only for  $k = c$ ), and therefore  $\frac{\partial r}{\partial c} > 0$ . Similarly,  $\frac{\partial r}{\partial c} < 0$  for  $c \geq K$ . Therefore, all (local) maxima of  $r(c)$  must lie in the interval  $(1, K)$ .

As  $\gamma_{mm,k} = \frac{g_m(k-c)^2 P_{m,k}}{N_m}$ ,  $r_k$  is symmetric around  $c = k$ . From (17),  $\frac{\partial^2 r_k}{\partial c^2}$  is negative for  $k - c = 0$ , and there exists a unique  $c_0$  at which  $\frac{\partial^2 r_k}{\partial c^2} = 0$  (this part can be shown by solving  $\gamma_{mm,k}(c)(2 + \gamma_{mb,k} + 3\gamma_{mm,k}(c)) - (1 + \gamma_{mb,k}) = 0$ , which is a quadratic equation in terms of  $(k - c)^2$  with a unique zero; see the proof of Lemma 7.5). For  $|k - c| > |k - c_0|$ ,  $\frac{\partial^2 r_k}{\partial c^2}$  is positive. This is true, e.g., for  $\gamma_{mm,k}(c) \geq 1$ .

Visually, each  $r_k$  as a function of  $c$  is a symmetric bell-shaped curve centered at  $k$ . Therefore,  $r$  can be seen as a sum of shifted and equally spaced symmetric bell-shaped curves. This sum, in general, can have linear in  $K$  number of local maxima. Examples with  $K$  local maxima can be constructed by choosing sufficiently large  $\frac{g_m P_{m,k}}{N_m}$  (sufficiently "narrow" bell-shaped curves).  $\square$

PROOF OF LEMMA 7.5. Assume that  $\gamma_{mm,k} > 0$  and  $\gamma_{mb,k} > 0 \forall k \in \{1, \dots, K\}$ , as otherwise  $\left| \frac{\partial r_k}{\partial c} \right| = 0$  and can be ignored.

**Case 1.** Assume first that  $c = k^*$  for some  $k^* \in \{1, \dots, K\}$ . Then, using (5),  $\frac{\partial r_k}{\partial c} = 0$ , and for every  $k \neq k^*$ :

$$\begin{aligned} \left| \frac{\partial r_k}{\partial c} \right| &\leq \frac{2}{\ln 2} \gamma_{mb,k} \frac{g_m P_m}{N_m} \frac{|k - c|}{(1 + \gamma_{mm,k}(c))(1 + \gamma_{mb,k})} \\ &\leq \frac{2}{\ln 2} \frac{\frac{g_m P_m}{N_m} |k - c|}{1 + \frac{g_m P_m}{N_m} (k - c)^2} \cdot \frac{\gamma_{mb,k}}{1 + \gamma_{mb,k}} \leq \frac{2}{\ln 2} \frac{1}{|k - c|}, \end{aligned}$$

since  $k - c \geq 1$ . Observe that since  $c = k^* \in \{1, \dots, K\}$ , every  $c - k$  is a positive integer. Therefore:

$$\begin{aligned} \left| \frac{\partial r}{\partial c} \right| &= \left| \sum_{k=1}^K \frac{\partial r_k}{\partial c} \right| \leq \frac{2}{\ln 2} \left| - \sum_{j=1}^{k^*-1} \frac{1}{|j - k^*|} + \sum_{k=k^*+1}^K \frac{1}{|k - k^*|} \right| \\ &\leq \frac{2}{\ln 2} \sum_{k=1}^{K-1} \frac{1}{k} = \frac{2}{\ln 2} H_{K-1}, \end{aligned}$$

where  $H_{K-1}$  is the  $(K - 1)^{\text{th}}$  harmonic number. Using the known inequality  $H_n < \ln(n) + 0.58 + \frac{1}{2n}$  for  $n \in \mathbb{N}$  [31] and

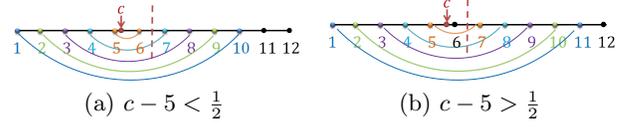


Figure 11: Pairing of points that are left and right from  $c$  for (a)  $c \in (5, 5.5)$  and (b)  $c \in (5.5, 6)$ .

assuming  $K \geq 4$ , we get:  $\left| \frac{\partial r}{\partial c} \right| < \frac{2}{\ln 2} (\ln(K) + 1)$ . For  $K < 4$ , by inspection:  $\sum_{k=1}^{K-1} \frac{1}{k} < \ln(K) + 1$ .

**Case 2.** Assume that  $c \notin \{1, \dots, K\}$ , and observe that for  $|k - c| \geq 1$ :  $\left| \frac{\partial r_k}{\partial c} \right| \leq \frac{2}{\ln 2} \frac{1}{|k - c|} \leq \frac{2}{\ln 2} \frac{1}{\lfloor |k - c| \rfloor}$ .

There can be at most two  $k$ 's with  $|k - c| < 1$ . For such  $k$ , we bound  $\left| \frac{\partial r_k}{\partial c} \right|$  as follows. First, observe from (16) and (17) that  $\frac{\partial}{\partial |k - c|} \left| \frac{\partial r_k}{\partial c} \right| = -\frac{\partial^2 r_k}{\partial c^2}$ . From (17),  $\frac{\partial^2 r_k}{\partial c^2} = 0$  if and only if for some  $c_0$ :

$$\begin{aligned} \gamma_{mm,k}(c_0)(2 + \gamma_{mb,k} + 3\gamma_{mm,k}(c_0)) - (1 + \gamma_{mb,k}) &= 0 \\ \Leftrightarrow \gamma_{mm,k}(c_0) &= \frac{(2 + \gamma_{mb,k}) + \sqrt{(2 + \gamma_{mb,k})^2 + 12(1 + \gamma_{mb,k})}}{6}. \end{aligned}$$

Note we have used that  $\gamma_{mm,k} > 0$  to get a unique solution for  $\gamma_{mm,k}$ . Since  $\gamma_{mm,k}(c_0) = \frac{g_m P_{m,k}}{N_m} (k - c_0)^2$ :

$$\begin{aligned} (k - c_0)^2 &= \frac{N_m}{g_m P_{m,k}} \gamma_{mm,k}(c_0) \\ &> \frac{N_m}{g_m P_{m,k}} \frac{2 \cdot (2 + \gamma_{mb,k})}{6} > \frac{N_m}{g_m P_{m,k}} \frac{\gamma_{mb,k}}{3}. \end{aligned}$$

From condition 7.4 we have that  $\frac{N_m}{g_m P_{m,k}} \cdot \gamma_{mb,k} \geq 1$ , which gives  $|k - c_0| > \frac{1}{\sqrt{3}}$ . It is clear from (5) and  $\gamma_{mm,k} = \frac{g_m P_{m,k}}{N_m} (k - c)^2$  that  $\frac{\partial^2 r_k}{\partial c^2}$  is negative for  $|k - c| < |k - c_0|$  and positive for  $|k - c| > |k - c_0|$ . Since  $\frac{\partial}{\partial |k - c|} \left| \frac{\partial r_k}{\partial c} \right| = -\frac{\partial^2 r_k}{\partial c^2}$ , it follows directly that  $\left| \frac{\partial r_k}{\partial c} \right|$  is maximized at  $c = c_0$ . Therefore, for  $|k - c| < 1$ , we have that  $\left| \frac{\partial r_k}{\partial c} \right| < \frac{2}{\ln 2} \frac{1}{|k - c_0|} < \frac{2}{\ln 2} \sqrt{3}$ .

Combining the results for  $|k - c| \geq 1$  and  $|k - c| < 1$ :

$$\begin{aligned} \left| \frac{\partial r}{\partial c} \right| &\leq \sum_{k=1}^K \left| \frac{\partial r_k}{\partial c} \right| \leq \frac{2}{\ln 2} \left( \left| - \sum_{j=1}^{\lfloor c \rfloor - 1} \frac{1}{|j - c|} + \sum_{k=\lfloor c \rfloor + 1}^K \frac{1}{|k - c|} \right| + 2\sqrt{3} \right) \\ &\leq \frac{2}{\ln 2} \left( \sum_{k=1}^{K-1} \frac{1}{k} + 2\sqrt{3} \right) < \frac{2}{\ln 2} (\ln(K) + 1 + 2\sqrt{3}). \end{aligned}$$

$\square$

PROOF OF LEMMA 7.9. From Lemma 7.8,  $P_{m,k}^* = \alpha_k P_m$ , where  $P_m = \bar{P}_m$ , and recalling that  $R_k = g_m \bar{P}_m (k - c)^2$ :

- $\alpha_k = \frac{\alpha_K \cdot (N_m + \alpha_K g_m P_m (K - c)^2)}{N_m}$  if  $k = c$ ;
- $\alpha_k = \frac{-N_m + \sqrt{N_m^2 + 4\alpha_K (N_m + \alpha_K g_m P_m (K - c)^2) g_m P_m (k - c)^2}}{2g_m P_m (k - c)^2}$  if  $k \neq c$ ;

and  $\alpha_K > 0$  is chosen so that  $\sum_{k=1}^K \alpha_k = 1$ . To simplify the notation, we will let  $\gamma_{mm} = \frac{g_m P_m}{N_m}$ , and write  $\alpha_k$  as:

$$\alpha_k = \begin{cases} \alpha_K \cdot (1 + \alpha_K \gamma_{mm} (K - c)^2), & \text{if } k = c, \\ \frac{-1 + \sqrt{1 + 4\alpha_K (1 + \alpha_K \gamma_{mm} (K - c)^2) \gamma_{mm} (k - c)^2}}{2\gamma_{mm} (k - c)^2}, & \text{if } k \neq c \end{cases} \quad (18)$$

Notice that for  $c = 1 + l \cdot \frac{1}{2}$ ,  $l \in \{1, 2, \dots, 2K - 3\}$ , the power allocation is symmetric around  $c$ , that is:  $\alpha_{\lfloor \frac{c}{2} \rfloor} = \alpha_{\lceil \frac{c}{2} \rceil}$ ,  $\alpha_{\lfloor \frac{c}{2} \rfloor - 1} = \alpha_{\lceil \frac{c}{2} \rceil + 1}$ , etc.

The first partial derivative of  $r$  with respect to  $c$  is:

$$\begin{aligned} \frac{\partial r}{\partial c} &= \sum_{k=1}^K \frac{\partial}{\partial c} \left( \log \left( \frac{1}{N_m(1 + \frac{g_m P_m}{N_m} \alpha_k (k-c)^2)} \right) \right) \\ &= \sum_{k=1}^K \frac{\partial}{\partial c} \left( \log \left( \frac{1}{1 + \gamma_{mm} \alpha_k (k-c)^2} \right) \right) \\ &= \sum_{k=1}^K \frac{2\gamma_{mm} \alpha_k (k-c)}{1 + \gamma_{mm} \alpha_k (k-c)^2} \end{aligned} \quad (19)$$

Observe that given the optimal power allocation (18):

- If  $c = \frac{K+1}{2}$ , then (from (18))  $\alpha_1 = \alpha_K$ ,  $\alpha_2 = \alpha_{K-1}, \dots$ ,  $\alpha_{\lfloor \frac{K+1}{2} \rfloor} = \alpha_{\lceil \frac{K+1}{2} \rceil}$ , and it follows that  $\frac{\partial r}{\partial c} = 0$ .
- If  $c = 1 + l \cdot \frac{1}{2}$ , for  $l \in \{0, 1, \dots, K-2\}$ , then, as  $\{\alpha_k\}$  is symmetric around  $c$ :  $\frac{\partial r}{\partial c} = \sum_{i=1}^{\lfloor c \rfloor} \frac{2\gamma_{mm} \alpha_i (i-c)}{1 + \gamma_{mm} \alpha_i (i-c)^2} + \sum_{j=\lfloor c \rfloor + 1}^{2\lfloor c \rfloor} \frac{2\gamma_{mm} \alpha_j (j-c)}{1 + \gamma_{mm} \alpha_j (j-c)^2} + \sum_{k=2\lfloor c \rfloor + 1}^K \frac{2\gamma_{mm} \alpha_k (k-c)}{1 + \gamma_{mm} \alpha_k (k-c)^2} > 0$ .
- If  $c = 1 + l \cdot \frac{1}{2}$ , for  $l \in \{K, \dots, K-2\}$ , then, as  $\{\alpha_k\}$  is symmetric around  $c$ :  $\frac{\partial r}{\partial c} = \sum_{i=1}^{2c-K-1} \frac{2\gamma_{mm} \alpha_i (i-c)}{1 + \gamma_{mm} \alpha_i (i-c)^2} + \sum_{j=2c-K}^{\lfloor c \rfloor} \frac{2\gamma_{mm} \alpha_j (j-c)}{1 + \gamma_{mm} \alpha_j (j-c)^2} + \sum_{k=\lfloor c \rfloor + 1}^K \frac{2\gamma_{mm} \alpha_k (k-c)}{1 + \gamma_{mm} \alpha_k (k-c)^2} < 0$ .

In other words, if we restrict our attention only to those  $\{\alpha_k\}$  that determine the optimal power allocation, then considering  $c$ 's from the set  $1 + l \cdot \frac{1}{2}$ , where  $l \in \{0, 1, \dots, 2K-2\}$ , we get that the first derivative of  $r$  with respect to  $c$  is positive for  $c < \frac{K+1}{2}$ , equal to zero for  $c = \frac{K+1}{2}$ , and negative for  $c > \frac{K+1}{2}$ . To conclude that at the global maximum for  $r$  we have  $c = \frac{K+1}{2}$  by considering  $c \in (1, K)$  it remains to show that for  $c \in (1 + l \cdot \frac{1}{2}, 1 + (l+1) \cdot \frac{1}{2})$ , where  $l \in \{0, 1, \dots, 2K-2\}$ , we have that  $\frac{\partial r}{\partial c} > 0$  if  $l \leq K-2$  and  $\frac{\partial r}{\partial c} < 0$  if  $l \geq K-1$ .

Fix any  $l \in \{0, 1, \dots, K-2\}$  (on the left half of the interval  $[1, K]$ ) and let  $c \in (1 + l \cdot \frac{1}{2}, 1 + (l+1) \cdot \frac{1}{2})$ . We make the following three claims:

- (K1) Each point  $i \in \{1, 2, \dots, \lfloor c \rfloor\}$  (left from  $c$ ) can be paired to a point  $j \in \{\lceil c \rceil, \lceil c \rceil + 1, \dots, K\}$  such that all the pairs are mutually disjoint and for each pair  $(i, j)$  we have that  $c - i < j - c$ .

*Proof of (K1):* To construct the pairing, observe that, by the choice of  $c$ ,  $c$  is between two consecutive integer points and is strictly closer to one of them. If it is closer to the left point, then the pairing is  $(\lfloor c \rfloor, \lceil c \rceil)$ ,  $(\lfloor c \rfloor - 1, \lceil c \rceil + 1), \dots, (1, 2\lfloor c \rfloor)$ . If  $c$  is closer to the right point, then the pairing is  $(\lfloor c \rfloor, \lceil c \rceil + 1)$ ,  $(\lfloor c \rfloor - 1, \lceil c \rceil + 2), \dots, (1, 2\lfloor c \rfloor + 1)$ . Such pairings must exist as  $c < \frac{K+1}{2}$ . The pairings for  $K = 12$  and cases:  $c \in (5, 5.5)$  and  $c \in (5.5, 6)$  are illustrated in Fig. 11. Q.E.D.

- (K2) In the optimal power allocation that corresponds to a given  $c$  and for any  $i, j \in \{1, \dots, K\}$ , if  $|i - c| < |j - c|$ , then  $\alpha_i > \alpha_j$ . In other words, the smaller the distance between  $k \in \{1, \dots, K\}$  and  $c$ , the larger the  $\alpha_k$ .

*Proof of (K2):* The proof has two parts. First, assume that  $|i - c| = 0$  and observe  $\alpha_j$  for  $|j - c| > 0$ . From (18):

$$\alpha_i = \alpha_K(1 + \alpha_K \gamma_{mm} (K - c)^2), \text{ and}$$

$$\begin{aligned} \alpha_j &= \frac{-1 + \sqrt{1 + 4\alpha_K(1 + \alpha_K \gamma_{mm} (K - c)^2) \gamma_{mm} (j - c)^2}}{2\gamma_{mm} (j - c)^2} \\ &= \frac{-1 + \sqrt{1 + 4\alpha_i \gamma_{mm} (j - c)^2}}{2\gamma_{mm} (j - c)^2}. \end{aligned}$$

Using simple algebraic transformations:

$$\begin{aligned} \alpha_j &< \alpha_i \\ &\Leftrightarrow \frac{-1 + \sqrt{1 + 4\alpha_i \gamma_{mm} (j - c)^2}}{2\gamma_{mm} (j - c)^2} < \alpha_i \\ &\Leftrightarrow \sqrt{1 + 4\alpha_i \gamma_{mm} (j - c)^2} < 1 + 2\alpha_i \gamma_{mm} (j - c)^2, \end{aligned}$$

we get that  $\alpha_j < \alpha_i$  by squaring both sides of the last term, as  $|j - c| > 0$  implies  $(2\alpha_i \gamma_{mm} (j - c)^2)^2 > 0$ .

Second, assuming that  $|k - c| > 0$  and taking the first derivative of  $\alpha_k$  with respect to  $(k - c)^2$ , we show that  $\alpha_k$  decreases as  $(k - c)^2$  (and consequently  $|k - c|$ ) increases. Let  $\Delta = (k - c)^2$ . Then, as:

$$\begin{aligned} \frac{d\alpha_k}{d\Delta} &= \frac{d}{d\Delta} \left( \frac{-1}{2\gamma_{mm} \Delta} + \frac{\sqrt{1 + 4\alpha_K(1 + \alpha_K \gamma_{mm} (K - c)^2) \gamma_{mm} \Delta}}{2\gamma_{mm} \Delta} \right) \\ &= \frac{1}{2\gamma_{mm} \Delta^2} - \frac{1 + 2\alpha_K(1 + \alpha_K \gamma_{mm} (K - c)^2) \gamma_{mm} \Delta}{2\gamma_{mm} \Delta^2 \sqrt{1 + 4\alpha_K(1 + \alpha_K \gamma_{mm} (K - c)^2) \gamma_{mm} \Delta}}, \end{aligned}$$

it follows that  $\frac{d\alpha_k}{d\Delta} < 0$ , since

$$\frac{\sqrt{1 + 4\alpha_K(1 + \alpha_K \gamma_{mm} (K - c)^2) \gamma_{mm} \Delta}}{1 + 2\alpha_K(1 + \alpha_K \gamma_{mm} (K - c)^2) \gamma_{mm} \Delta} < 1. \text{ Q.E.D.}$$

- (K3) As  $|k - c|$  increases,  $\left| \frac{\partial r_k}{\partial c} \right| = \frac{2\gamma_{mm} \alpha_k |k - c|}{1 + \gamma_{mm} \alpha_k (k - c)^2}$  decreases.

*Proof of (K3):* Observe that:

$$\frac{\partial}{\partial \alpha_k} \left| \frac{\partial r_{i,k}}{\partial c} \right| = \frac{2\gamma_{mm} |k - c|}{(1 + \gamma_{mm} \alpha_k (k - c)^2)^2} > 0.$$

We had from (K2) that  $\frac{d\alpha_k}{d|k - c|} < 0$ , and therefore:

$$\frac{\partial}{\partial |k - c|} \left| \frac{\partial r_k}{\partial c} \right| = \frac{\partial}{\partial \alpha_k} \left| \frac{\partial r_k}{\partial c} \right| \cdot \frac{d\alpha_k}{d|k - c|} < 0, \text{ Q.E.D.}$$

Using (19), we can write  $\frac{\partial r_i}{\partial c}$  as:

$$\begin{aligned} \frac{\partial r_i}{\partial c} &= \sum_{k=1}^K \frac{2\gamma_{mm} \alpha_k (k - c)}{1 + \gamma_{mm} \alpha_k (k - c)^2} \\ &= \sum_{i=1}^{\lfloor c \rfloor} \frac{2\gamma_{mm} \alpha_i (i - c)}{1 + \gamma_{mm} \alpha_i (i - c)^2} + \sum_{j=\lfloor c \rfloor + 1}^K \frac{2\gamma_{mm} \alpha_j (j - c)}{1 + \gamma_{mm} \alpha_j (j - c)^2}. \end{aligned}$$

If  $c \in [1, \frac{K+1}{2})$ , then, from (K1), each term  $i$  in the left summation can be paired to a term  $j$  in the right summation, such that all the pairs are disjoint and for each pair  $(i, j)$ :  $|i - c| < |j - c|$ . From (K3), for each such pair  $(i, j)$ :  $\frac{2\gamma_{mm} \alpha_i |i - c|}{1 + \gamma_{mm} \alpha_i (i - c)^2} < \frac{2\gamma_{mm} \alpha_j |j - c|}{1 + \gamma_{mm} \alpha_j (j - c)^2}$ . As all the terms in the left summation are negative, and all the terms in the right summation are positive, it follows that:

$$\begin{aligned} \frac{\partial r}{\partial c} &= \sum_{i=1}^{\lfloor c \rfloor} \frac{2\gamma_{mm} \alpha_i (i - c)}{1 + \gamma_{mm} \alpha_i (i - c)^2} + \sum_{j=\lfloor c \rfloor + 1}^K \frac{2\gamma_{mm} \alpha_j (j - c)}{1 + \gamma_{mm} \alpha_j (j - c)^2} \\ &= - \sum_{i=1}^{\lfloor c \rfloor} \frac{2\gamma_{mm} \alpha_i |i - c|}{1 + \gamma_{mm} \alpha_i (i - c)^2} + \sum_{j=\lfloor c \rfloor + 1}^K \frac{2\gamma_{mm} \alpha_j |j - c|}{1 + \gamma_{mm} \alpha_j (j - c)^2} > 0. \end{aligned}$$

Proving that  $\frac{\partial r}{\partial c} < 0$  for  $c \in (\frac{K+1}{2}, K]$  is symmetrical to the proof that  $\frac{\partial r}{\partial c} > 0$  for  $c \in [1, \frac{K+1}{2})$ . As  $\frac{\partial r}{\partial c} = 0$  for  $c = \frac{K+1}{2}$ , at the globally maximum  $r$  we have that  $c = \frac{K+1}{2}$ .  $\square$