An Asymptotic Probabilistic Analysis of Vehicle Routing

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1 Introduction

Recently, Simchi-Levi and Bramel (1990) analyzed an interesting probability model of the following routing problem. Consider a set of points $p_i = (x_i, y_i)$, $0 \leq i \leq n$, in the plane, where p_0 denotes the location of a depot containing goods/material to be delivered to the $n \geq 1$ customers located at the remaining points. At the depot there is a fleet of delivery vehicles, all having the same capacity. The demand of the customer at p_i is given by a number $\mu_i = \mu(p_i)$, $0 < \mu_i \leq 1$ that denotes the fraction of a vehicle's capacity needed by the delivery to this customer.

A *routing* for a given problem instance is a partition of the n customer locations such that for all blocks B in the partition, $\sum_{p_i \in B} \mu(p_i) \leq 1$. The partition has the interpretation that all customers with locations in the same block are served by the same vehicle. We assume that a vehicle serves its customers by following a minimum-length tour (roundtrip) from the depot. The length of a routing is the sum of its constituent tour lengths. Assuming that the supply of vehicles is unlimited, the problem is to find a minimum-length routing.

Let H denote a heuristic that generates approximate solutions to this NP-complete routing problem. Then for a given depot location p_0, $H_n \equiv H(\{p_i, \mu_i\}_{i=1}^n)$ denotes the length of the routing produced by H. Assume that the locations (x_i, y_i), $1 \leq i \leq n$, are i.i.d. random variables with compact support in $\mathbb{R}^2$, and with an expected distance $E(d) < \infty$ from the depot. Simchi-Levi and Bramel (1990) give a constructive argument based on classical matching theory which shows that, if the μ_i are n independent samples of a uniform random variable on $[0, 1]$, then there exists a heuristic H such that with probability 1

$$\lim_{n \to \infty} \frac{H_n}{n} = E(d) \ .$$

This paper shows that much stronger results are possible if a uniform distribution also applies to the customer locations. For example, suppose the p_i, $1 \leq i \leq n$, are chosen independently and uniformly at random throughout some rectangular region of the plane. Then as shown in Section 3,

$$E[OPT_n] = nE[d] + \Theta(n^{2/3}) \ ,$$

where OPT denotes an optimal routing policy. Moreover, an efficient heuristic H is defined which achieves this bound and hence is optimal up to a constant factor in the expected absolute error $E[H_n] - nE[d]$.

2 Preliminaries

We begin by stating a well-known result that will be useful in proving a lower bound. Classical estimates of the binomial distribution prove

Theorem 2.1 *Let n points be chosen independently and uniformly at random in the unit cube $[0,1]^3$. The expected distance between a point and its nearest neighbor is $\Theta(n^{-1/3})$.*

Remarks. This result holds under several standard generalizations of the assumptions. For example, the number of points can be αn for any fixed $\alpha > 0$; the expectation can be limited to a fixed, positive fraction of the points; and the unit cube can be replaced by any cuboid of fixed dimensions $a, b, c > 0$. Also, the number of points can be Poisson distributed with mean n. In all cases, only the hidden multiplicative constant in $\Theta(n^{-1/3})$ is affected. ■

We put instances of the routing problem within the setting of Theorem 2.1 as follows. A customer is represented by a labeled point in 3 dimensions, with the first two coordinates (x, y) giving the customer's location. If the customer's demand is $\mu \leq 1/2$ then the point's third coordinate is $z = \mu$ and its label is a minus; otherwise, the third coordinate is $z = 1 - \mu$ and the point's label is a plus.

We assume that demand is uniformly distributed on $[0, 1]$, so the z coordinate is uniform on $[0, 1/2]$ and the sign of a point is equally likely to be a plus or minus. For convenience, we take the unit square $0 \leq x, y \leq 1$ as the normalized, square area containing customer locations. Thus in our probability model, to be called the *uniform model*, the n customer points are i.i.d. uniform random samples from $[0, 1]^2 \times [0, 1/2]$, with labels being independent and equally likely to be plus or minus.

In the above setting, matching algorithms in 3 dimensions can be proposed as routing heuristics that limit tours to at most two customer locations. For example, the Upward Matching (UM) heuristic determines a maximum matching of plus points to minus points such that in each matched pair the minus point is below the plus point, i.e., has a smaller z coordinate. UM then assigns a vehicle (tour) to each matched pair of customers and one to each unmatched customer. The resulting routing is valid because if a plus point (x_i, y_i, z_i) is matched to a minus point (x_j, y_j, z_j) then $z_j \leq z_i$ ensures that the capacity demands $1 - z_i$ and z_j sum to at most one.

The ordered matching (OM) heuristic[†] is like UM except that a plus point (x_i, y_i, z_i) matched to a minus point (x_j, y_j, z_j) is above the minus point in *all* coordinates, i.e., $x_i \geq x_j$, $y_i \geq y_j$, and $z_i \geq z_j$. OM may have limited practical use, because the ordering on the x and y coordinates may not be needed for a valid routing. However, the following result on ordered matching will be useful in proving an upper bound.

Theorem 2.2 (Karp, Luby, and Marchetti-Spaccamela (1984)). *Let n points be chosen independently and uniformly at random in $[0,1]^3$, and suppose that each point is labeled a plus point or minus point with equal probability, independently of the remaining points. In a maximum ordered matching of plus points to minus points, the expected number of unmatched points is $\Theta(n^{2/3})$.*

3 Asymptotic Bounds

The following notation is needed. For a given depot location p_0, P_0 denotes a set of n labeled customer points (x_i, y_i, z_i), $1 \leq i \leq n$. The distance in the (x, y) plane between the customer locations corresponding to points $p, p' \in P_0$ is denoted by $d(p, p')$. If one of these points, say p', is the depot, then the notation is abbreviated to $d(p)$. The functional notation $x(p)$, $y(p)$, and $z(p)$ denotes the corresponding coordinate values of p.

The main result follows.

Theorem 3.1 *In the uniform model of the routing problem,*

$$E[OPT_n] \; = \; nE[d] \; + \; \Theta(n^{2/3}) \; .$$

Proof. We prove the upper and lower bounds separately, starting with the comparatively easy upper bound.

Upper bound: We prove $E[OPT_n] = nE[d] + O(n^{2/3})$. In what follows, p^+ and p^- denote generic plus and minus points in instances P_0 of the routing problem.

Recall the ordered-matching routing heuristic defined in Section 2. We need only prove that

$$E[OM_n] \; = \; nE[d] + O(n^{2/3}) \; .$$

Consider an ordered matching of an instance P_0 of the routing problem. Let $M = M(n)$ denote the set of matched pairs (p^+, p^-) and let $U = U(n)$ denote the set of unmatched points that remain. The length of the OM routing can be expressed as

$$OM_n \; = \; \sum_{p \in P_0} d(p) \; + \sum_{(p^+, p^-) \in M} d(p^+, p^-) \; + \; \sum_{p \in U} d(p) \; . \tag{3.1}$$

[†] In two dimensions, ordered matching is called up-right matching.

The expected value of the first sum is $nE[d]$. By Theorem 2.2, we have $E|U| = O(n^{2/3})$, so for the third sum,

$$\sum_{p \in U} E[d(p)] \;=\; O(n^{2/3}) \;.$$

It remains to verify that $E[X] = O(n^{2/3})$, where X is the second sum in (3.1). By the triangle inequality,

$$X \leq \sum_{(p^+,p^-)\in M} [x(p^+) - x(p^-)] \;+\; \sum_{(p^+,p^-)\in M} [y(p^+) - y(p^-)] \;. \tag{3.2}$$

By symmetry,

$$\begin{aligned} E\left[\sum_{(p^+,p^-)\in M} [x(p^+) - x(p^-)]\right] &= E\left[\sum_{p^- \in U} x(p^-)\right] - E\left[\sum_{p^+ \in U} x(p^+)\right] \\ \sum_{(p^+,p^-)\in M} E[y(p^+) - y(p^-)] &= E\left[\sum_{p^+ \in U} y(p^-)\right] - E\left[\sum_{p^+ \in U} y(p^+)\right] \;. \end{aligned} \tag{3.3}$$

By Theorem 2.2 again, $E|U| = O(n^{2/3})$, so after substituting (3.3) into (3.2), we obtain the desired result, $E[X] = O(n^{2/3})$. ■

Lower Bound: To prove

$$E[OPT_n] = nE(d) + \Omega(n^{2/3}) \;, \tag{3.4}$$

the following notation will be useful. Let S denote the unit square in the (x, y) plane. For any given sample, the arguments below often focus on the set $P = P(n)$ of locations in S with customers having demands in $(1/3, 2/3)$. Note that at most two points from P can be visited in the same tour. Let $M = M(n)$ denote the set of pairs (p^+, p^-), $p^+, p^- \in P$, such that p^+ and p^- are in the same tour, along with perhaps other points not in P. The remaining minus and plus points of P that are not in any pair of M make up the set $U = U(n)$. The subsets of plus and minus points are denoted respectively by U^+ and U^-, $U^+ \bigcup U^- = U$.

The location of the depot is arbitrary (either inside or outside S), but we will frequently need to consider subsets of points whose distances from the depot satisfy some strictly positive lower bound. To this end, it is convenient to define a vertical third, S_*, of S such that $d(p) \geq 1/6$ for all $p \in S_*$. In particular, define $S_* = [0, 1] \times [0, 1/3]$ if $x_0 \geq 1/2$ and $S_* = [0, 1] \times [2/3, 1]$ if $x_0 < 1/2$, as shown in Fig. 1. P_* denotes the subset of points in S_*. Replacing P by P_* in the definitions of M and U yields sets denoted by M_* and U_*. U_*^+ and U_*^- are defined in analogy with U^+ and U^-. Note that every minus point in P_* is accounted for either in U_*^- or in some pair of M_*. The following simple lower bounds will prove useful.

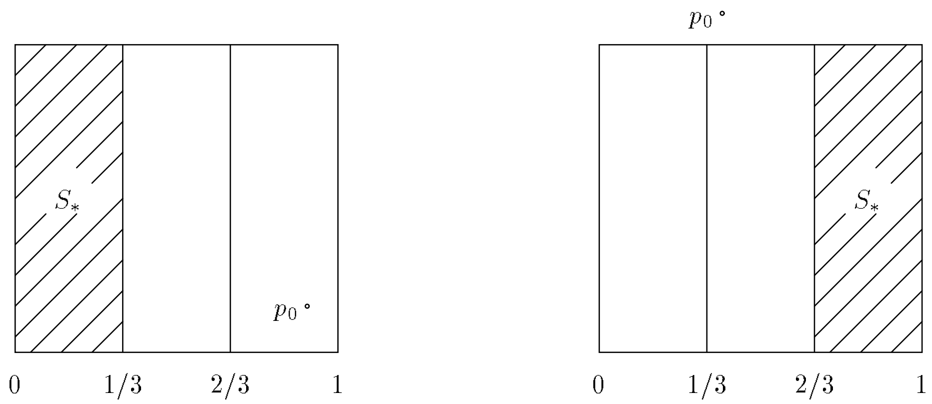

Figure 1: Examples of S_*; p_0 is the depot location.

For a given instance P_0, the lower bound

$$E[OPT_n] \geq E\left[\sum_{p^+ \in P_0} 2d(p^+)\right] = nE[d] \tag{3.5}$$

is obvious, since there are at least as many tours as there are plus points, and since the length of a tour is at least twice the distance from the depot to any point on the tour.

A tour visiting no plus points has a length at least the sum of the distances $p^- \in P_*$ in the tour. This follows because each such p^- has a demand exceeding 1/3, so at most two such p^- can appear in the same tour. Then, since (3.5) does not account for tours without plus points, (3.5) can be tightened to

$$E[OPT_n] \geq nE[d] + E\left[\sum_{p^- \in U^-} d(p^-)\right] \tag{3.6}$$

Finally, note that the length of any tour visiting a $p^+ \in P_0$ and a $p^- \in P_0$ is at least $d(p^+) + d(p^-) = 2d(p^+) + d(p^-) - d(p^+)$. Thus, if the sum of $d(p^-) - d(p^+)$ over some random set of such tours is positive on average, then (3.5) can be improved to

$$E[OPT_n] \geq nE[d] + E\left[\sum_{(p^+, p^-) \in Q} d(p^-) - d(p^+)\right] , \tag{3.7}$$

where Q is the random set containing the pairs (p^+, p^-) in the given set of tours.

The proof now proceeds by analyzing two cases, depending on whether or not $E|M_*|$ is a constant fraction of $|P_*| = 2E|M_*| + E|U_*| = n/9$ for all n sufficiently large.

Case 1: $E|M_*| = o(n)$ as $n \to \infty$. Clearly, in this case $E|U_*| = \Omega(n)$ and hence $E|U_*^-| = \Omega(n)$. Let S_{**} be the vertical half of S_* farthest from the depot, as illustrated in Fig. 2, and let P_{**} be the set of points in S_{**}. Now $E|M_*| = o(n)$ also implies that the expected number of minus points in P_{**} sharing tours with plus points in P_* is $o(n)$. But the expected number of minus points in P_{**} is $\Omega(n)$, so we must have an average of $\Omega(n)$ minus points $p^- \in P_{**}$ with one of the properties:

1. p^- is in a tour with no plus points,
2. p^- is in a tour with a plus point not in P_*, i.e., in $P - P_*$.

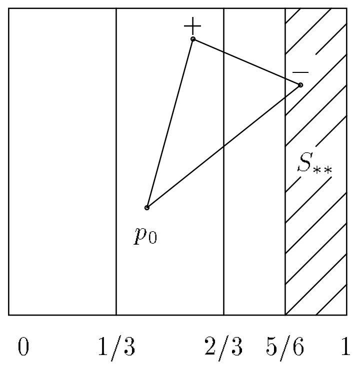

Figure 2: Example of S_{**} and a tour visiting a $p^+ \in P - P_*$ and a $p^- \in P_{**}$.

Let U^-_{**} be the set of minus points in P_{**} with property 1, and let M_{**} be the set of pairs (p^-, p^+), $p^- \in P_{**}$, $p^+ \in P - P_*$, with p^- and p^+ in the same tour. Then one or both of $E|U^-_{**}| = \Omega(n)$ and $E|M_{**}| = \Omega(n)$ must hold. Now $U^-_{**} \subseteq U^-$, so if $E|U^-_{**}| = \Omega(n)$, then (3.4) follows from (3.6) and the bound $d(p) \geq 1/3$, $p \in P_{**}$. Thus, assume that $E|M_{**}| = \Omega(n)$. In a tour visiting a minus point in P_{**} and a plus point in $P - P_{**}$, the minus point is a distance at least 1/3 from the depot and 1/6 from the plus point (see Fig. 2). It is easy to see that there is then an $\alpha > 0$ such that the length of any such tour is at least $2d(p^+) + \alpha$, where $p^+ \in P - P_*$ is the plus point in the tour. Terms $2d(p^+)$ are already accounted for in (3.5), so adding the lower bound α for each pair in M_{**} yields

$$E[OPT_n] \geq E\left[\sum_{p^+ \in P_0} 2d(p^+)\right] + \alpha E|M_{**}| \ .$$

Then $E|M_{**}| = \Omega(n)$ proves (3.4).

Case 2: Assume now that $E|M_*| = \Omega(n)$. The distance between p^+, $p^- \in M_*$ in the 3 dimensions of the problem instance is denoted by $d^{(3)}(p^+, p^-)$. By the triangle inequality, we have

$$d^{(3)}(p^+, p^-) \leq z(p^+) - z(p^-) + d(p^+, p^-) \ ,$$

where we have used the fact that $z(p^+) \geq z(p^-)$ for all p^+, $p^- \in M_*$, since $\mu(p^+) + \mu(p^-) = 1 - z(p^+) + z(p^-) \leq 1$. By $E|M_*| = \Omega(n)$ and Theorem 2.1 (see also the remarks following the theorem), it is easily verified that

$$E\left[\sum_{(p^+,p^-)\in M_*} z(p^+) - z(p^-)\right] + E\left[\sum_{(p^+,p^-)\in M_*} d(p^+,p^-)\right] \geq \qquad E\left[\sum_{(p^+,p^-)\in M_*} d^{(3)}(p^+,p^-)\right] = \Omega(n^{2/3}) \ . \tag{3.8}$$

Then one or both of the first two expectations in (3.8) is $\Omega(n^{2/3})$. Consider each separately.

Case 2.1: Suppose that

$$E\left[\sum_{(p^+,p^-)\in M_*} z(p^+) - z(p^-)\right] = \Omega(n^{2/3}) \ . \tag{3.9}$$

Enumerate the pairs in M_* and let $\chi_i(t) = 1$, for $z(p_i^-) \leq t \leq z(p_i^+)$ and $\chi_i(t) = 0$, otherwise, where (p_i^+, p_i^-) is the i^{th} pair of M_*. Then $\int_{1/3}^{1/2} \chi_i(t)dt = z(p_i^+) - z(p_i^-)$, so a summation on i yields

$$\sum_{(p^+,p^-)\in M_*} [z(p^+) - z(p^-)] = \int_{1/3}^{1/2} \sum_{1\leq i\leq |M_*|} \chi_i(t)dt = \int_{1/3}^{1/2} m_*(t)dt \ , \tag{3.10}$$

where $m_*(t)$ is the number of pairs $(p^+, p^-) \in M_*$ such that $z(p^-) \leq t \leq z(p^+)$. In the uniform model it is easy to see that (3.9) and (3.10) imply

Proposition 3.1 *There exists a β, $1/3 < \beta < 1/2$, such that there is an average of $E[m_*(\beta)] = \Omega(n^{2/3})$ tours visiting a $p^+ \in P_*$ and a $p^- \in P_*$ with $\mu(p^+) \leq 1 - \beta$ and $\mu(p^-) \leq \beta$.*

For a β satisfying Proposition 3.1, let $P_\beta \subseteq P$, be the set of points p with demands satisfying $\beta \leq \mu(p) \leq 1 - \beta$. Define M_β as the set of pairs $(p^+, p^-) \in M$ with both $p^+ \in P_\beta$ and $p^- \in P_\beta$. The remaining points in P_β form the set U_β, with the subsets U_β^+ and U_β^- of plus and minus points, respectively. Note that points in U_β^- cannot share a tour with a plus point; either a plus point has a demand in $[1-\beta, 2/3]$, which is too large, or it is excluded by definition because its demand is in $[1/2, 1-\beta]$. Hence, $U_\beta^- \subseteq U^-$.

By symmetry, $E\left[\sum_{p^-\in P_\beta} d(p^-)\right] - E\left[\sum_{p^+\in P_\beta} d(p^+)\right] = 0$. It is convenient to break down the sums and rearrange as follows:

$$E\left[\sum_{\substack{(p^+,p^-)\in M_\beta \\ d(p^-)>d(p^+)}} d(p^-) - d(p^+)\right] \; - \; E\left[\sum_{\substack{(p^+,p^-)\in M_\beta \\ d(p^+)\geq d(p^-)}} d(p^+) - d(p^-)\right] \; + \; E\left[\sum_{p^-\in U_\beta^-} d(p^-)\right] - E\left[\sum_{p^+\in U_\beta^+} d(p^+)\right] = 0 \ . \tag{3.11}$$

By Proposition 3.1 and $d(p) \geq 1/6$, $p \in P_*$, the expected value of the last of the four sums is $\Omega(n^{2/3})$. Then either the first or third expected value must be $\Omega(n^{2/3})$. If the first were $\Omega(n^{2/3})$, then (3.4) would follow from (3.7), with Q defined as $\{(p^+, p^-) \in M_\beta | d(p^-) > d(p^+)\}$; and if the third were $\Omega(n^{2/3})$, then since $U_\beta^- \subseteq U^-$, (3.4) would follow from (3.6).

Case 2.2: Suppose that the second expected value in (3.8) is $\Omega(n^{2/3})$, and hence

$$E\left[\sum_{(p^+,p^-)\in M} d(p^+, p^-)\right] = \Omega(n^{2/3}) \ . \tag{3.12}$$

Let V be the subset of pairs (p^+, p^-) in M such that

$$d(p^-) + d(p^+, p^-) - d(p^+) \geq \frac{1}{2} d(p^+, p^-) \tag{3.13}$$

or

$$d(p^+) - d(p^-) \leq \frac{1}{2} d(p^+, p^-) \ .$$

The remaining pairs (p^+, p^-) of M form the set $W = M - V$ with

$$d(p^+) - d(p^-) > \frac{1}{2} d(p^+, p^-) \ . \tag{3.14}$$

Then (3.12) becomes

$$E\left[\sum_{(p^+,p^-)\in V} d(p^+, p^-)\right] + E\left[\sum_{(p^+,p^-)\in W} d(p^+, p^-)\right] = \Omega(n^{2/3}) \ . \tag{3.15}$$

At least one of the two expected values in (3.15) must be $\Omega(n^{2/3})$; each is considered independently below.

Case 2.2.1: Assume that

$$E\left[\sum_{(p^+,p^-)\in V} d(p^+, p^-)\right] = \Omega(n^{2/3}) \ . \tag{3.16}$$

By (3.13), the length of a tour containing a pair in V is bounded by

$$d(p^+) + d(p^-) + d(p^+, p^-) \geq 2d(p^+) + \frac{1}{2} d(p^+, p^-) \ .$$

The distances $2d(p^+)$ are already accounted for in (3.5), so adding the distances $\frac{1}{2}d(p^+, p^-)$, $(p^+, p^-) \in V$, to (3.5) gives

$$E[OPT_n] \geq nE[d] + E\left[\sum_{(p^+,p^-)\in V} \frac{1}{2} d(p^+, p^-)\right] \ . \tag{3.17}$$

Then the lower bound (3.4) follows from (3.16) and (3.17).

Case 2.2.2: Assume that

$$E\left[\sum_{(p^+,p^-)\in W} d(p^+,p^-)\right] = \Omega(n^{2/3}) \ . \tag{3.18}$$

Recall that U^+ and U^- are the sets of plus and minus points, respectively, not in any pair of M. Let V^+ and V^- be the respective sets of plus and minus points in pairs of V; W^+ and W^- are defined analogously with respect to W. Note that all of the plus points and minus points in P are accounted for in U^+, V^+, W^+ and U^-, V^-, W^-, respectively. Trivially,

$$\begin{aligned} E\left[\sum_{p^-\in U^-} d(p^-)\right] \ + \ E\left[\sum_{p^-\in V^-} d(p^-)\right] \ + \ E\left[\sum_{p^-\in W^-} d(p^-)\right] & \\ = E\left[\sum_{p^+\in U^+} d(p^+)\right] \ + \ E\left[\sum_{p^+\in V^+} d(p^+)\right] \ + \ E\left[\sum_{p^+\in W^+} d(p^+)\right] & \ , \end{aligned}$$

so

$$E\left[\sum_{p^-\in U^-} d(p^-)\right] + E\left[\sum_{(p^+,p^-)\in V} d(p^-) - d(p^+)\right] \geq E\left[\sum_{(p^+,p^-)\in W} d(p^+) - d(p^-)\right] \ . \tag{3.19}$$

Then by (3.14) and (3.18), the right-hand side of (3.19) gives the estimate

$$E\left[\sum_{p^-\in U^-} d(p^-)\right] + E\left[\sum_{(p^+,p^-)\in V} d(p^-) - d(p^+)\right] = \Omega(n^{2/3}) \ . \tag{3.20}$$

At least one of these expected values must be $\Omega(n^{2/3})$. But if the first is $\Omega(n^{2/3})$, then (3.4) follows from (3.6); and if the second is $\Omega(n^{2/3})$, then (3.4) follows from (3.7) with $Q = V$. ∎

References

Karp, R. M., M. Luby, and A. Marchetti-Spaccamela (1984), "Probabilistic Analysis of Multidimensional Bin Packing Problems," *Proc. 16th ACM Symp. Th. Comput.*, ACM Press, pp. 289-298.

Simchi-Levi, D. and J. Bramel (1990), "On the Optimal Solution Value of the Capacitated Vehicle Routing Problem with Unsplit Demands," Working Paper, Dept. Industrial Engineering and Operations Research, Columbia University, New York, NY 10027.

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ABSTRACT

Consider a set of points $p_i = (x_i, y_i)$, $0 \leq i \leq n$, in the plane, where p_0 denotes the location of a depot containing goods/material to be delivered to the $n \geq 1$ customers located at the remaining points. An unlimited number of vehicles, all having the same capacity, is located at the depot. The demand at p_i is given by a number $0 < \mu_i \leq 1$ that denotes the fraction of a vehicle's capacity needed by the delivery to the customer at p_i. A routing for a given problem instance is a partition of the n customer locations such that for all blocks B in the partition the demands at points in B can be satisfied by a single vehicle, i.e., $\sum_{p_i \in B} \mu_i \leq 1$. All customers with locations in the same block are served by the same vehicle following a minimum-length tour (roundtrip) from the depot. The problem is to find a partition that minimizes the total length of the routing, i.e., the sum of the constituent tour lengths.

For a given depot location p_0, let $OPT_n = OPT(\{p_i, \mu_i\}_{i=1}^n)$ denote the length of an optimal routing for this NP-complete problem. Assume that the μ_i's are independently and uniformly distributed on $[0,1]$ and that the p_i are independently and uniformly distributed in the unit square. We prove that, under this uniform model,

$$E[OPT_n] = nE[d] + \Theta(n^{2/3}) \ ,$$

where $E[d]$ is the expected distance from the depot to the customers. In addition, we present an efficient heuristic H that achieves this bound and hence is optimal up to a constant factor in the expected absolute error $E[H_n] - nE[d]$.