Packing Random Intervals On-Line

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1 Introduction

Let $(a_1, b_1), \ldots, (a_n, b_n)$ be a sequence of n independent random intervals in $\mathbf{R}_+$ to be packed on-line in a given interval $[0, x]$; an interval $I_i = (a_i, b_i)$ is packed (marked as occupied in $[0, x]$) if and only if $I_i \subseteq [0, x]$ and I_i does not overlap any interval in $I_1, \ldots, I_{i-1}$ that has already been packed. Suppose the I_i are drawn independently from a given interval distribution. To avoid trivialities, we assume that the distribution is such that $I_i \subseteq [0, x]$ with probability 1. The *on-line interval packing problem* asks for the expectation of the number $N_x(n)$ of intervals packed as a function of n and the parameters of the interval distribution.

The asymptotic behavior of the expected number packed on-line was found in [4] for the case where, for each i, a_i and b_i are taken as the smaller and larger of two independent random draws from $[0, x]$. In this case, x is only a scale factor and does not figure in the result. The asymptotic expected number packed is given by

$$\mathrm{E}N_x(n) \sim c n^{(\sqrt{17}-3)/4}$$

as $n \to \infty$, where $c \approx 0.98$ approximates the value of an explicit formula given in [4].

In this paper, by a completely different analysis, we obtain the corresponding result for the case where the a_i are independent uniform random draws from $[0, x-1]$ and all intervals have unit lengths ($b_i = a_i + 1$). Now, the result clearly depends on x; indeed, we prove the following limit law for the expected number packed: If for some constant $\lambda > 0$, we put $n = \lfloor \lambda x \rfloor$, then

$$\lim_{x \to \infty} \frac{\mathrm{E}N_x(\lfloor \lambda x \rfloor)}{x} = \alpha(\lambda), \tag{1}$$

where

$$\alpha(\lambda) = \int_{v=0}^{\lambda} \exp\left(-2 \int_0^v \frac{1 - e^{-x}}{x} dx\right) dv. \tag{2}$$

We note in passing that the probabilistic analysis of optimal *off-line* interval packing has been worked out in [10], where the number packed is to be maximized, and in [5], where the unoccupied space of the packing is to be minimized. Also, for the worst-case analysis of combinatorial models of on-line interval packing, see [8, 11].

Our interest in on-line interval packing originated in a stochastic version, where the arrival times and left endpoints of the intervals to be packed form a two-dimensional Poisson process with a mean normalized to 1 per unit time per unit distance. In this version, which we will call the *Poisson version*, $K(t,x)$ denotes the expected number packed during $[0,t]$ into $[0,x]$, and we prove that, as might be expected, $K(t,x)$ has a property similar to that in (1),

$$K(\lambda x, x) \sim \alpha(\lambda) x \tag{3}$$

as $x \to \infty$ with λ fixed, where $\alpha(.)$ is given by (2). As we will verify, the Poissonization of the input length makes the analysis easier; we then obtain (1) from (3) via the strong law of large numbers.

The classical parking problem of Renyi [16] has an intimate relationship with on-line interval packing. In the former problem, unit-length cars are parked sequentially along a curb (interval) $[0,x]$, $x > 1$. Each car chooses a parking place independently and uniformly at random from those available, i.e., from those where it will not overlap cars already parked or the curb boundaries. Thus, the left endpoint of the first car is a uniform random draw from $[0, x-1]$, and if the first car is parked in $[y, y+1]$, then the left endpoint of the second car parked is drawn uniformly at random from $[0, y-1]$, $[0,y-1]\cup[y+1,x-1]$, or $[y+1,x-1]$ according as $y > x-2$, $1 \le y \le x-2$, or $y < 1$, respectively. This uniform parking of cars continues until every unoccupied gap is less than 1 in length, i.e., no further cars can be parked. Renyi [16] showed that the mean of $\tilde{N}_x$, the number parked at the conclusion of the process, satisfies $\mathrm{E}\tilde{N}_x \sim \alpha_* x$, $x \to \infty$, with the *Renyi constant* $\alpha_* = \lim_{\lambda\to\infty} \alpha(\lambda)$ determined by (2). This result actually follows from a more refined estimate of $\mathrm{E}N_x$ which was later improved by Dvoretsky and Robbins [6]. (The more precise estimate is given in Section 2 for comparison with our new results.) For many other improvements and extensions of the results, see [6, 7, 12, 13, 14] and the references therein.

Although the parking process differs from the on-line (unit) interval packing process, it is easy to verify that, given n cars (unit intervals) already parked (packed) in $[0,x]$, the conditional joint distribution of the $n+1$ gaps is the same under both processes. Moreover, we expect (and will prove) that

$$\lim_{n\to\infty} \lim_{x\to\infty} \frac{\mathrm{E}N_x(n)}{x} = \lim_{x\to\infty} \frac{\mathrm{E}\tilde{N}_x}{x} = \alpha_*. \tag{4}$$

Thus, in the limit, the fraction of the positive real line that is occupied is the same under both processes.

Mackenzie [12] showed that the asymptotic variance is

$$\mathrm{Var}(\tilde{N}_x) \sim \mu_* x, \tag{5}$$

where, in terms of $\tilde{\alpha}(y) := \alpha_* - \alpha(y)$, and

$$\beta(y) := \exp\left(-2\int_0^y \frac{1-e^{-y}}{y} dy\right), \tag{6}$$

the constant is given by

$$\begin{aligned}\mu_* &= 4\int_0^\infty \left[e^{-y}(1-e^{-y})\frac{\tilde{\alpha}(y)}{y} - \frac{e^{-2y}(e^{-y}-1+y)\tilde{\alpha}^2(y)}{\beta(y)y^2}\right] dy - \alpha_*, \\ &= 0.03815\ldots.\end{aligned} \tag{7}$$

For the Poisson model of on-line interval packing, the variance of the number packed in $[0, x]$ during $[0, t]$ presents an interesting contrast; in Section 3 we prove a detailed estimate which implies that, for a function $\mu(t)$ depending only on t,

$$\mathrm{Var}^2(N_x(t)) \sim 4\alpha(t)\beta(t)x \log x + \mu(t)x + o(x), \tag{8}$$

as $x \to \infty$. We have been unable to find a probabilistic argument that accounts for the presence of a $\log x$ factor in the leading term of (8) but not (5). On the other hand, we will point out how the analysis creates this difference in the two asymptotic results. Note that $\beta(t) \to 0$, as $t \to \infty$, and that $\mu(t) \to \mu_*$, as $t \to \infty$, must hold since

$$\lim_{t\to\infty} \frac{\mathrm{Var}(N_x(t))}{x} = \mu_*$$

follows from the fact that the on-line interval packing problem becomes the parking problem in the limit $t \to \infty$.

We conclude this section with a brief discussion of applications. There have been many applications of the parking problem in the physical sciences, including space-filling problems, molecular adsorption on surfaces, order-disorder theory, and problems in the theory of liquids (see [12, 15, 16]). We expect that the on-line interval packing problem is also worth considering in these settings, but our interest stems from the scheduling problems of existing and proposed multimedia communication systems. The application of this type cited in [4] arises in a one-dimensional loss network (modeled by the interval $[0, 1]$) where the intervals to be packed represent calls between pairs of communicating stations (points in $[0, 1]$).

Modeling reservation protocols in communication systems was the source of our stochastic version of on-line interval packing (see e.g., [9] and the references therein). In a baseline reservation model, there is a single resource and there are randomly arriving requests , each specifying a future time interval during which it wants to use the resource. A request arriving at time t identifies its interval of usage $[t_1, t_2]$ by giving the advance notice $t_1 - t$ and the duration $t_2 - t_1$. Scheduling decisions are made on-line: a requested reservation is approved/accepted if and only if the specified interval does not overlap an interval already reserved for some earlier request. Now consider the following stochastic set-up: requests are Poisson arrivals at rate λ, advance notices are independently and uniformly distributed over $[0, a]$ for some given a, and intervals have unit durations. Suppose that, at some time t in equilibrium, we look at the pattern of unit intervals that were reserved during $[t - x, t]$ for some large x. If a is large relative to x, one expects that, except for negligible edge effects, this pattern is approximately the same stochastically as the pattern of intervals packed in $[0, x]$ by the stochastic version of on-line interval packing during the time interval $[0, \lambda]$. This statement is made rigorous and a corresponding limit law proved in [3]; certain generalizations of the above model are also accommodated.

2 Expected Number Packed

In Section 2.1, we begin with the Poisson version and derive a first-order estimate of the expected number of intervals packed in $[0, x]$ during $[0, t]$, which we have denoted by $K(t, x)$. We then show how this result can be used to prove (1), the corresponding result for the model with fixed-length input. In Section 2.2, we prove a much more precise estimate for $K(t, x)$.

2.1 Proof of (1)

Consider the Poisson model and note that $K(t, x) = 0$ if $0 \leq x < 1$, since no interval of length 1 can fit into $[0, x]$ in that case. As in Renyi's analysis in one dimension, we will compute a Laplace transform in the space dimension, eventually to obtain a first-order partial differential equation for

$$\mathcal{K}(t, u) := \int_0^\infty K(t, x) e^{-ux} dx, \tag{9}$$

which can be solved by standard methods.

Consider the behavior of K in a time interval $[t, t + \Delta t]$ with $x \geq 1$. Arriving intervals with left endpoints in $(x-1, x]$ are rejected, so if no interval with its left endpoint in $[0, x-1]$ arrives at some time in $[t, t+\Delta t]$, then nothing happens and $K(t+\Delta t, x) = K(t, x)$. On the other hand, if an interval $[y, y+1]$ arrives and $y \in [0, x-1]$, then $K(t + \Delta t, x)$ will be the sum of $K(t, y)$ and $K(t, x-y-1)$ plus 1 for the interval packed. Thus, we have

$$K(t+\Delta t, x) = [1-(x-1)\Delta t] K(t, x) + \Delta t \left[\int_0^{x-1} \left(K(t, y) + K(t, x-y-1) + 1 \right) dy \right], \tag{10}$$

ignoring terms of order $o(\Delta t)$ (which correspond to the probabilities of events in which multiple arrivals occur). By symmetry, the integral simplifies and the above becomes

$$K(t + \Delta t, x) = [1 - (x-1)\Delta t] K(t, x) + \Delta t \left[2 \int_0^{x-1} K(t, y) dy + x - 1 \right], \tag{11}$$

which yields

$$\begin{aligned} \frac{\partial K}{\partial t}(t, x) &= -(x-1) K(t, x) + 2 \int_0^{x-1} K(t, y) dy + x - 1 \\ &= (x-1)[1 - K(t, x)] + 2 \int_0^{x-1} K(t, z) dz, \end{aligned} \tag{12}$$

for $t \geq 0$ and $x \geq 1$. Now multiply both sides by e^{-ux} and integrate with respect to x over $[1, \infty)$. Exploiting the fact that $K(t, x) = 0,\ 0 \leq x < 1$, we get from (9)

$$\frac{\partial \mathcal{K}}{\partial t} = \frac{e^{-u}}{u^2} + \mathcal{K} + \frac{\partial \mathcal{K}}{\partial u} + 2 \int_1^\infty e^{-ux} dx \int_0^{x-1} K(t, z) dz. \tag{13}$$

Next, reverse the order of integration in (13) so that the integral becomes $\int_{z=0}^\infty \int_{x=z+1}^\infty (.) dz dx$. Carrying out the integration gives $2(e^{-u}/u)\mathcal{K}$ for the last term in (13), so (13) can be written

$$\frac{\partial \mathcal{K}}{\partial t} = \frac{e^{-u}}{u^2} + (1 + \frac{2e^{-u}}{u})\mathcal{K} + \frac{\partial \mathcal{K}}{\partial u} \tag{14}$$

For the boundary conditions, observe first that $K(t,x) \leq x$ for all t. This bound along with $K(0,x) = 0$ implies the boundary conditions

$$\mathcal{K}(0,u) = 0, \quad \mathcal{K}(t,\infty) = 0. \tag{15}$$

The solution to the first-order partial differential equation (13) that satisfies (15) can be determined by standard methods. In our case, we simply input (13) into *Mathematica*©, and found that

$$\mathcal{K}(t,u) = e^{-2\mathrm{Ei}(-u)} \int_{v=0}^{t} e^{-u+2\mathrm{Ei}(-u-v)} \frac{dv}{(u+v)^2}, \tag{16}$$

where $\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^y}{y} dy,\ x \neq 0$, is the exponential integral, with the Cauchy principal value of the integral understood when $x > 0$. However, it is also easy to obtain (16) directly, especially since (14) can be reduced to an ordinary differential equation with the change of variables $v = u + t,\ w = t$. To put (16) into a form that we can check against Renyi's result in the limit $t \to \infty$, we rewrite (16) as

$$\begin{aligned} \mathcal{K}(t,u) &= \int_{v=0}^{t} e^{-u+2(\mathrm{Ei}(-u-v)-\mathrm{Ei}(-u))} \frac{dv}{(u+v)^2} \\ &= e^{-u} \int_{v=0}^{t} \exp\left(-2\int_{u}^{u+v} \frac{e^{-x}}{x} dx\right) \frac{dv}{(u+v)^2} \qquad (17) \\ &= \frac{e^{-u}}{u^2} \int_{v=0}^{t} \exp\left(-2\int_{u}^{u+v} \frac{1-e^{-x}}{x} dx\right) dv, \qquad (18) \end{aligned}$$

where the integrands are now entire functions.

At this point, we can easily prove

Theorem 1 *The fraction of* $\mathbf{R}_+$ *that is occupied after time* t *is given by*

$$\lim_{x\to\infty} \frac{K(t,x)}{x} = \alpha(t) = \int_{v=0}^{t} \exp\left(-2\int_{0}^{v} \frac{1-e^{-x}}{x} dx\right) dv. \tag{19}$$

Remark. In the limit $t \to \infty$, no remaining gap in the final, absorbing state of $\mathbf{R}_+$ is large enough to accommodate an interval, so we obtain (4) for the Poisson model.

Proof. From (18) it easily follows that

$$\mathcal{K}(t,u) \sim \frac{\alpha(t)}{u^2} \quad \text{as} \quad u \searrow 0. \tag{20}$$

Now, since $\int_0^x K(t,z)dz$ is monotonically increasing in x, application of Karamata's Tauberian theorem [2] p. 37, yields

$$\int_0^x K(t,z)dz \sim \alpha(t)\frac{x^2}{2} \quad as \quad x \to \infty, \tag{21}$$

where $\alpha(t)$ is the same as in (19). Next, since x is the expected number of arrivals per unit time in $[0,x]$, we have $\frac{\partial}{\partial t}K(t,x) \leq x$, which implies in particular that

$$\frac{\frac{\partial}{\partial t}K(t,x)}{x^2} \to 0, \quad as \quad x \to \infty. \tag{22}$$

Now divide (12) by x^2, let $x \to \infty$, and substitute (21) and (22) to obtain

$$K(t,x) \sim \alpha(t)x \quad as \quad x \to \infty. \tag{23}$$

∎

Let us now return to the problem with fixed-length input.

Corollary 2 *The limit in (1) holds for the on-line interval packing problem when intervals are unit-length and have left endpoints independently and uniformly drawn from* $[0, x-1]$.

Proof: We may express $K(\lambda, x+1)$ as an average of $N_x(X)$ with X Poisson distributed

$$K(\lambda, x+1) = \sum_{n=0}^{\infty} e^{-\lambda x}\frac{(\lambda x)^n}{n!}N_x(n). \tag{24}$$

Let S_x be the sum of $\lfloor x \rfloor + 1$ Poisson random variables, the first $\lfloor x \rfloor$ with parameter λ and the last one with parameter $(x - \lfloor x \rfloor)\lambda$. Choose a small $\epsilon > 0$; break up the sum in (24) into three partial sums, the middle one straddling the mean; and apply elementary bounds to obtain

$$\begin{aligned} K(\lambda, x+1) &\le \sum_{n=0}^{\lfloor(\lambda-\epsilon)x\rfloor} e^{-\lambda x}\frac{(\lambda x)^n}{n!}\cdot n + \sum_{n=\lfloor(\lambda-\epsilon)x\rfloor+1}^{\lfloor(\lambda+\epsilon)x\rfloor} e^{-\lambda x}\frac{(\lambda x)^n}{n!}N_x(\lfloor(\lambda+\epsilon)x\rfloor) \\ &\quad + \sum_{\lfloor(\lambda+\epsilon)x\rfloor+1}^{\infty} e^{-\lambda x}\frac{(\lambda x)^n}{n!}\cdot n \\ &\le N_x(\lfloor(\lambda+\epsilon)x\rfloor)\mathbf{P}(\lfloor(\lambda-\epsilon)x\rfloor \le S_x \le \lfloor(\lambda+\epsilon)x\rfloor) + \lfloor\lambda x\rfloor\mathbf{P}(S_x \le \lfloor(\lambda-\epsilon)x\rfloor) \\ &\quad + \lambda x\mathbf{P}(S_x \ge \lfloor(\lambda+\epsilon)x\rfloor), \end{aligned}$$

so by the strong law of large numbers, we get

$$K(\lambda, x+1) \le N_x(\lfloor(\lambda+\epsilon)x\rfloor)(1+o(1)) + o(1),$$

as $x \to \infty$, and

$$\underline{\lim}_{x\to\infty}\frac{N(\lfloor\lambda x\rfloor, x)}{x} \ge \lim_{x\to\infty}\frac{K(\lambda-\epsilon, x+1)}{x} = \alpha(\lambda-\epsilon) \nearrow \alpha(\lambda)$$

as $\epsilon \to 0$.

Similarly, for a lower bound, we find

$$\begin{aligned} K(\lambda, x+1) &\ge N_x(\lfloor(\lambda-\epsilon)x\rfloor)\mathbf{P}(\lfloor(\lambda-\epsilon)x\rfloor \le S_x \le \lfloor(\lambda+\epsilon)x\rfloor) \\ &= N_x(\lfloor(\lambda-\epsilon)x\rfloor)(1+o(1)), \end{aligned}$$

as $x \to \infty$, and so

$$\overline{\lim}_{x\to\infty}\frac{N_x(\lfloor\lambda x\rfloor)}{x} \le \alpha(\lambda+\epsilon) \searrow \alpha(\lambda),$$

as $\epsilon \to 0$, which completes the proof of the corollary. ∎

It is interesting to note that we can in fact prove (1) without direct reference to the stochastic version, although this seems much easier to see in hindsight. We start with a recurrence

$$\mathrm{E}N_{x+1}(n) = \frac{2}{x}\sum_{j=0}^{n-1}\binom{n-1}{j}\int_0^x \left(\frac{y-1}{x}\right)^j \left(\frac{x-y+1}{x}\right)^{n-j-1} \mathrm{E}N_y(j)dy + 1, \tag{25}$$

which expresses the fact that $\mathrm{E}N_{x+1}(n)$ is one plus the sum of the expected numbers packed to the left and right of the first interval packed; the factor of 2 comes from symmetry and the binomial law describes the number ($\leq n-1$) packed to the left of the first-packed interval $[y, y+1]$. At this point, we view the expected value in (24) as a transform, apply it to (25), and after a fair amount of algebra, we obtain (12). The remainder of the analysis is as before.

2.2 A Refined Estimate

In principle, extending the approach of Dvoretsky and Robbins [6], we can compute exact formulas for $K(t,x)$, $0 \leq x \leq j$, inductively as we increase $j = 1, 2, \ldots$. To this end, differentiate the function

$$f(t,x) := e^{t(x-1)}K(t,x), \tag{26}$$

to obtain

$$\frac{\partial K}{\partial t}(t,x) = e^{-t(x-1)}\frac{\partial f}{\partial t}(t,x) - (x-1)K(t,x),$$

and then substitute into (12) to get

$$\frac{\partial f}{\partial t}(t,x) = (x-1)e^{t(x-1)} + 2e^{t(x-1)}\int_0^{x-1} e^{-t(z-1)}f(t,z)dz. \tag{27}$$

We have the boundary condition $f(0,x) = K(0,x) = 0$, so an integration of (27) yields

$$f(t,x) = e^{t(x-1)} - 1 + 2\int_0^t e^{v(x-1)}dv \int_0^{x-1} e^{-v(z-1)}f(v,z)dz,$$

whereupon substitution of (26) gives

$$K(t,x) = 1 - e^{-t(x-1)} + 2e^{-t(x-1)}\int_0^t e^{v(x-1)}dv\int_0^{x-1} K(v,z)dz. \tag{28}$$

Thus, by (28) and the trivial fact that $K(t,x) = 0$, $0 \leq x < 1$, we compute
($1 \leq x \leq 2$):

$$K(t,x) = 1 - e^{-t(x-1)}, \tag{29}$$

noting that the limit $t \to \infty$ is 1, as it should be, and
($2 \leq x \leq 3$):

$$K(t,x) \;=\; 1 - e^{-t(x-1)} + 2e^{-t(x-1)}\int_0^t dv\int_1^{x-1} e^{v(x-1)}(1 - e^{-v(z-1)})dz$$

$$
\begin{aligned}
&= 1 - e^{-t(x-1)} + 2e^{-t(x-1)} \int_0^t \left[\frac{e^v - e^{v(x-1)}}{v} + (x-2)e^{v(x-1)} \right] dv \\
&= 3 - \frac{2}{x-1} - 2e^{-t(x-1)} \left[\frac{x-2}{x-1} - \int_0^t \frac{e^v - e^{v(x-1)}}{v} dv \right], \qquad (30)
\end{aligned}
$$

the sum $3-2/(x-1)$ agreeing, as it must, in the limit $t \to \infty$ with the corresponding parking result of Dvoretsky and Robbins [6, Equations (2.8) and (2.9)].

With the growth in complexity shown for the values $j = 1, 2, 3$, one is well advised to develop useful asymptotics. For the parking problem, the following refined estimate of $\mathrm{E}\tilde{N}_x$ is well known. There exists a constant $\theta > 0$ such that, as $x \to \infty$

$$
\mathrm{E}\tilde{N}_x = \alpha_* x + \alpha_* - 1 + O(e^{-\theta x \log x}) \qquad (31)
$$

Renyi [16] proved this result with a larger error term (viz., $O(1/x^m)$ for every m); Dvoretsky and Robbins [6] supplied the tighter error term. The estimate (31) is the limit $t \to \infty$ of the estimate below for the Poisson model of on-line (unit) interval packing, the main result of this section.

Theorem 3 *There exists a $\theta > 0$ such that, as $x \to \infty$ for any fixed t,*

$$
K(t, x) = \alpha(t)x + \alpha(t) + \beta(t) - 1 + O(e^{-\theta x \log x}). \qquad (32)
$$

Before giving the proof, we briefly discuss our approach and prove two lemmas. To this point, u has been treated as real, but for the remainder of the section it is to be taken as complex. For simplicity, we will suppress t and write $\mathcal{K}(u) = \mathcal{K}(t, u)$, t then being any fixed non-negative constant. A similar convention will apply to $\alpha = \alpha(t)$ and $\beta = \beta(t)$.

The inversion formula for Laplace transforms gives, for $x > 0$,

$$
K(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{K}(u) e^{xu} du, \qquad (33)
$$

the integration path being the vertical line $\Re u = c$, where c is any positive constant. The main idea in the proof of Theorem 3 is to shift this integration path as far left as possible and to use the Cauchy residue theorem to deal with singularities encountered in this shift. To do this, we need to study the analytic continuation of $\mathcal{K}(u)$ and to obtain a growth estimate for $|\mathcal{K}|$.

Lemma 4 *The function $\mathcal{K}(u)$ is analytic for all $u \neq 0$. At $u = 0$, $\mathcal{K}(u)$ has a second order pole and the expansion*

$$
\mathcal{K}(u) = \frac{\alpha}{u^2} + \frac{\alpha + \beta - 1}{u} + \cdots, \qquad (34)
$$

where α and β are as given in (19) and (6) .

Proof: Note that $\alpha(u)$ and $\beta(u)$ are entire functions of the complex variable u, so if we rewrite (18) as

$$
\mathcal{K}(u) = \frac{e^{-u}}{u^2} \cdot \frac{1}{\beta(u)} \int_0^t \beta(u+v) dv = \frac{e^{-u}}{u^2 \beta(u)} [\alpha(u+t) - \alpha(u)], \qquad (35)
$$

we see that $\mathcal{K}(u)$ is analytic for $u \neq 0$. The expansions of $e^{-u}/\beta(u)$ and $\alpha(u+t)-\alpha(u)$ about $u=0$ are given by

$$\begin{aligned}
\frac{e^{-u}}{\beta(u)} &= e^{-u+2\int_0^u (1-y/2+\cdots)dy} = e^{u-u^2/2+\cdots} \\
&= 1+u+0\cdot u^2+\cdots
\end{aligned}$$

and

$$\begin{aligned}
\alpha(u+t)-\alpha(u) &= [\alpha(t)-\alpha(0)]+[\beta(t)-\beta(0)]u+\cdots \\
&= \alpha(t)+[\beta(t)-1]u+\cdots .
\end{aligned}$$

Multiplying these gives the expansion in (34). ∎

Remark: We have shown that (18) gives the analytic continuation of $\mathcal{K}(u)$ to all $u \neq 0$. Equation (17) also continues $\mathcal{K}(u)$ analytically to the complex plane, but the exception here is the slit $[-t,0]$ (we interpret $\int_u^{u+t}(e^{-y}/y)dy$ as the integral of e^{-y}/y over the line segment $[-t,0]$).

Lemma 5 *Let $u \neq 0$, and let $\sigma := \Re u$ be any real number. Then*

$$|\mathcal{K}(u)| \leq \frac{C(\sigma)}{|u|^2}, \tag{36}$$

where $C(\sigma)$ denotes a positive constant depending only on σ.

Proof: Use the elementary estimates

$$\begin{aligned}
|e^{-u}| &\leq e^{-\sigma} \quad \text{for} \quad \Re u \geq \sigma, \\
|u+v| &\geq |u|/2 \quad \text{for} \quad |u| \geq 2t,\ 0 \leq v \leq t, \\
|e^{-y}/y| &\leq e^{-\sigma}/t \quad \text{for} \quad |u| \geq 2t,\ \Re u \geq \sigma,\ y \in [u, u+t],
\end{aligned}$$

to obtain

$$\left|\exp\left(2\int_u^{u+v}(e^{-y}/y)dy\right)\right| \leq \exp\left(\left|2\int_u^{u+v}(e^{-y}/y)dy\right|\right) \leq e^{2e^{-\sigma}},\ |u| \geq 2t,\ \Re u \geq \sigma,\ 0 \leq v \leq t,$$

so that, from (17),

$$|\mathcal{K}(u)| \leq \frac{4t\exp(-\sigma+2e^{-\sigma})}{|u|^2},\ |u| \geq 2t,\ \Re u \geq \sigma. \tag{37}$$

By Lemma 4, $\mathcal{K}(u)u^2$ is entire, so there exists a positive constant $C_1(t)$ depending only on t such that

$$|\mathcal{K}(u)| \leq C_1(t)/u^2 \quad \text{for} \quad 0 < u \leq 2t. \tag{38}$$

The lemma follows from (37) and (38) by choosing

$$C(\sigma) = \max[C_1(t), 4t\exp(-\sigma+2e^{-\sigma})].$$

∎

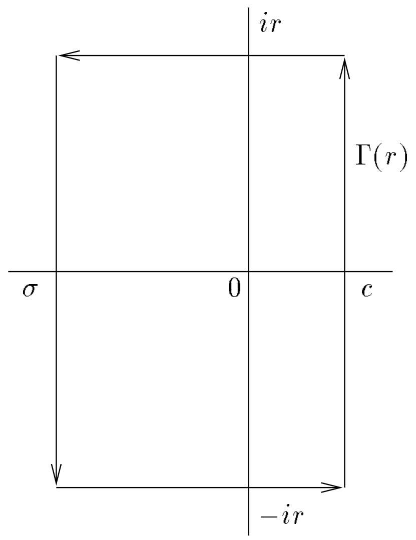

Figure 1: Rectangular contour of integration.

Proof of Theorem 3: For simplicity, we begin by proving the theorem with the weaker error estimate: $O(e^{-\xi x})$ for all $\xi > 0$. We shift the vertical integration path in (33) to the left of 0 and apply the residue theorem. First observe that, by Lemma 4, $\mathcal{K}(u)e^{xu}$ is analytic for all u, except for a pole at $u = 0$. From (34) and the power series expansion $e^{xu} = 1 + xu + \cdots$, we conclude that

$$\mathcal{K}(u)e^{xu} = \frac{\alpha}{u^2} + \frac{\alpha x + \alpha + \beta - 1}{u} + \cdots, \quad u \neq 0. \tag{39}$$

For any $\sigma > 0$, apply the residue theorem to the rectangular contour $\Gamma(r)$ sketched in Fig. 1, where r is any positive real. As $\mathcal{K}(u)$ is analytic on and inside $\Gamma(r)$, except for a pole at $u = 0$, we get from (39)

$$\frac{1}{2\pi i}\int_{\Gamma(r)} \mathcal{K}(u)e^{xu}du = \alpha x + \alpha + \beta - 1. \tag{40}$$

By Lemma 5, the contribution of the integral along the horizontal sides of $\Gamma(r)$ tends to 0 as $r \to \infty$, so (40) becomes

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{K}(u)e^{xu}du = \alpha x + \alpha + \beta - 1 + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{K}(u)e^{xu}du. \tag{41}$$

But if $u = -\xi + iy,\ -\infty < y < \infty,\ \xi > 0$, then by Lemma 5,

$$\left|\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{K}(u)e^{xu}du\right| \leq \frac{C(\xi)e^{-\xi x}}{2\pi}\int_{-\infty}^{\infty} \frac{dy}{\xi^2 + y^2} = \frac{C(\xi)}{2|\xi|}e^{-\xi x}, \tag{42}$$

and so $K(t,x) = \alpha(t)x + \alpha + \beta - 1 + O(e^{-\xi x})$ for all $\xi > 0$ follows from (41) and (42). Finally, the $O(e^{-\xi x})$ error term can be improved to $O(e^{-\theta x \log x})$ for any $0 < \theta < 1$ by using (37) and maximizing over $\sigma > 0$ for any fixed x. This is a routine calculus problem, so we omit the details. ∎

3 Second Moment

Before turning to the Poisson model, we give, again for contrast, a more precise estimate for the parking problem than was given in the Introduction.

Theorem 6 (Mackenzie [12] and Dvoretsky and Robbins [6]) *With μ_* given by (7), there exists a constant $\theta > 0$ such that the variance of the number parked in the parking problem satisfies*

$$\mathrm{Var}(\tilde{N}_x) = \mu_* x + \mu_* + O(e^{-\theta x \log x}) \tag{43}$$

Remark: Dvoretsky and Robbins [6] proved (43) without deriving an explicit formula for μ_*. Earlier, by computing the continuous limit of a result derived in a discretized version of the parking problem, Mackenzie [12] obtained formula (7) for μ_*. ∎

The combined analysis of Mackenzie and of Dvoretsky and Robbins gives an awkward, roundabout proof of the variance estimate which is unnecessary. A direct proof can be based on a careful analysis of an inverse Laplace transform, as in the previous section. A sketch of our new proof of (refVarNx) is given in the appendix; the reader may wish to delay reading the proof until after finishing this section, which uses a similar, albeit more involved approach to finding the variance of $N_x(t)$ in the Poisson model of on-line interval packing.

The main result of this section is the following estimate for the second moment $M(t, x) := \mathrm{E}N_x^2(t)$.

Theorem 7 *For every fixed t, there exist constants $\zeta = \zeta(t)$, $\phi = \phi(t)$, and $\psi_j = \psi_j(t)$ ($j \geq 0$) such that, as $x \to \infty$,*

$$M(t, x) = \alpha^2 x^2 + 4\alpha\beta x \log x + \phi x + \zeta \log x + \sum_{j=0}^{J} \psi_j / x^j + O(1/x^{J+1}), \tag{44}$$

for every integer $J \geq 0$.

Remarks: The forthcoming analysis could be used to derive expressions for ϕ, ζ, and the ψ_j, but they would be cumbersome and uninteresting, so we leave them unspecified. In view of (32), we have for the variance

$$\mathrm{Var}(N_x(t)) = 4\alpha\beta x \log x + \mu x + \zeta \log x + \sum_{j=0}^{J} \psi_j' / x^j + O(1/x^{J+1}), \tag{45}$$

where $\mu = \phi - 2\alpha(\alpha + \beta - 1)$ is as given in (8), and where $\psi_0' = \psi_0 - (\alpha + \beta - 1)^2$ and $\psi_j' = \psi_j$, $j > 0$. Recall that $\mu = \mu(t) \to \mu_*$ as $t \to \infty$, and note that the error term is less than that proved for $\mathrm{Var}(\tilde{N}_x)$. ∎

As in the previous section, before proving (44), we compute the transform in the space dimension

$$\mathcal{M}(t, u) := \int_0^\infty M(t, x) e^{-xu} dx. \tag{46}$$

After doing this in Section 3.1, we prove some of the analytic properties of $M(t,u)$ in Section 3.2. In Section 3.3, we dispose of some further preliminaries and then complete the proof of Theorem 7 in Section 3.4.

3.1 The Transform $\mathcal{M}(t,u)$

The derivation of the formula for $\mathcal{M}(t,u)$ duplicates that for $\mathcal{K}(t,u)$, so we will be brief. In analogy with (10), we have

$$M(t+\Delta t) = \mathrm{E}N_x^2(t+\Delta t) = [1-(x-1)\Delta t]M(t,x)+$$

$$\Delta t\int_0^{x-1} \mathrm{E}[1+N_y^2(t)+N_{x-y-1}^2(t)+2N_y(t)+2N_{x-y-1}(t)+2N_y(t)+2N_{x-y-1}(t)]dy+o(t),$$

and hence

$$\begin{aligned}\frac{\partial M}{\partial t}(t,x) &= (x-1)[1-M(t,x)]+ \qquad (47)\\ &\quad 2\int_0^{x-1} M(t,y)dy+4\int_0^{x-1} K(t,y)dy+2\int_0^{x-1} K(t,y)K(t,x-y-1)dy,\end{aligned}$$

the analogue of (12). Now take the transform of both sides of (47) (see (46)) and obtain

$$\frac{\partial \mathcal{M}}{\partial t}(t,u) = \frac{\partial \mathcal{M}}{\partial u}(t,u)+\mathcal{M}(t,u)\left(1+\frac{2e^{-u}}{u}\right)+\frac{e^{-u}}{u^2}\mathcal{A}(t,u),\ t\geq 0,\ u>0, \qquad (48)$$

where

$$\mathcal{A}(t,u) = 1+4u\mathcal{K}(t,u)+2(u\mathcal{K}(t,u))^2, \qquad (49)$$

with the boundary conditions

$$\mathcal{M}(0,u)=0,\quad \mathcal{M}(t,\infty)=0,$$

as before. Solving (47) as we did (14), we find

$$\mathcal{M}(t,u) = e^{-u}\int_0^t \frac{\mathcal{A}(t,u+v)}{(u+v)^2}\exp\left(2\int_u^{u+v}\frac{e^{-y}}{y}dy\right)dv \qquad (50)$$

which is equivalent to

$$\mathcal{M}(t,u) = \frac{e^{-u}}{u^2}\int_0^t \mathcal{A}(t,u+v)\exp\left(-2\int_u^{u+v}\frac{1-e^{-y}}{y}dy\right)dv. \qquad (51)$$

3.2 Analytic Properties of $\mathcal{M}(t,u)$

For the remainder of the section we suppress the dependence on t, which is to be taken as fixed, unless otherwise noted. We first establish the analytic properties of $\mathcal{A}(u)$ and $\mathcal{B}(u) := \mathcal{A}(u)\beta(u)$.

Lemma 8 *(i) $\mathcal{A}(u)$ is analytic for $u \neq 0$ and has a second-order pole at $u = 0$ along with the expansion*

$$\mathcal{A}(u) = \frac{2\alpha^2}{u^2} + \frac{4\alpha(\alpha+\beta)}{u} + \cdots . \tag{52}$$

(ii) $\mathcal{B}(u)$ is analytic for $u \neq 0$ and has a second-order pole at $u = 0$ along with the expansion

$$\mathcal{B}(u) = \frac{2\alpha^2}{u^2} + \frac{4\alpha\beta}{u} + \cdots . \tag{53}$$

Proof: Part (i) follows from (35) and (49). For part (ii), note that $\beta(u)$ is entire and that its expansion at $u = 0$ is given by

$$\begin{aligned}\beta(u) &= \exp\left(-2\int_0^u \frac{1-e^{-y}}{y}dy\right) = e^{-2\int_0^u (1+y/2+\cdots)dy} \\ &= e^{-2u+u^2+\cdots} = 1 - 2u + \cdots .\end{aligned} \tag{54}$$

Now multiply (52) by (54) to obtain (53). ∎

Lemma 9 *The transform $\mathcal{M}(u)$ has an analytic continuation to $\mathbf{R}$ with continuous boundary values on $I^o := (-t, 0)$, the boundary values being distinct on the two sides of I^o.*

Proof: The analytic continuation of $\mathcal{M}(u)$ to $\mathbf{R}$ is given by (51), which we rewrite as

$$\mathcal{M}(u) = \frac{e^{-u}}{u^2\beta(u)} \int_u^{u+t} \mathcal{B}(v)dv. \tag{55}$$

Now let $\mathcal{D}(u) := \mathcal{B}(u) - 2\alpha^2/u^2 - 4\alpha\beta/u$, and note that, by part (ii) of Lemma 8, $\mathcal{D}(u)$ is entire. Let $\mathbf{R}_1$ be the complex plane from which we have removed the negative real axis $(-\infty, 0]$. From (55), we get

$$\begin{aligned}\mathcal{M}(u) &= \frac{e^{-u}}{u^2\beta(u)} \int_u^{u+t} \left(\mathcal{D}(v) + \frac{2\alpha^2}{v^2} + \frac{4\alpha\beta}{v}\right) dv \\ &= \mathcal{M}_1(u) + \mathcal{M}_2(u)\log u, \quad u \in \mathbf{R}_1,\end{aligned} \tag{56}$$

where

$$\mathcal{M}_1(u) = \frac{e^{-u}}{\beta(u)u^2}\left[\int_u^{u+t} \mathcal{D}(v)dv + 2\alpha^2\left(\frac{1}{u} - \frac{1}{u+t}\right) + 4\alpha\beta\log(u+t)\right] \tag{57}$$

$$\mathcal{M}_2(u) = -\frac{4\alpha\beta}{u^2}. \tag{58}$$

In (56) and (57), $\log u$ and $\log(u+t)$ denote principal values of the logarithm, i.e., $|\Im \log u|, |\Im \log(u+t)| < \pi$ for $u \in \mathbf{R}_1$. Formulas (57) and (58) show that $\mathcal{M}_1(u)$ is analytic for $\Re u > -t$, except for a third-order pole at $u = 0$, and that $\mathcal{M}_2(u)$ is analytic for all u except for a second-order pole at $u = 0$. The statement in the lemma concerning boundary values is then evident from formula (56). ∎

At $u = 0$, we have the expansions

$$\mathcal{M}_1(u) = \sum_{j=-3}^{\infty} A_j u^j, \quad \mathcal{M}_2(u) = \sum_{j=-2}^{\infty} B_j u^j. \tag{59}$$

The coefficients A_j and B_j can be computed but they become progressively more complicated as j increases. In the sequel, we only require

$$A_{-3} = 2\alpha^2, \quad B_{-2} = -4\alpha\beta. \tag{60}$$

Next, we need an estimate of $|\mathcal{M}(u)|$.

Lemma 10 *Let σ be any real number. For $\Re u > 0$ and $|u| > 2t$, we have*

$$|\mathcal{M}(u)| \leq \left[1 + \frac{4C(\sigma)}{t} + 2\left(\frac{C(\sigma)}{t}\right)^2\right] \frac{C(\sigma)}{|u|^2}, \tag{61}$$

where $C(\sigma)$ is as defined in Lemma 5.

Proof: For $|u| > 2t$ and $0 \leq v \leq t$, we have $|u+v| \geq t$. ¿From this observation and (36), we conclude that

$$|\mathcal{A}(u+v)| \leq 1 + \frac{4C(\sigma)}{t} + 2\left(\frac{C(\sigma)}{t}\right)^2, \quad |u| > 2t,\ 0 \leq v \leq t. \tag{62}$$

Using (62), the proof of Lemma 10 follows that of Lemma 5. ∎

3.3 Further Preliminaries

We begin by converting the inverse transform for $M(x) = M(t, x)$ into a more useful integral formula. This result is followed by three technical lemmas that will finally put us in position to complete the proof of Theorem 7 in Section 3.4.

Let Γ denote the contour shown in Fig. 2 as consisting of five parts:

Γ_1 and Γ_5 are the vertical half-lines from $-a - i\infty$ to $-a$ and $-a$ to $-a + i\infty$,
Γ_2 and Γ_4 are the two horizontal lines from $-a$ to $-r$ and $-r$ to $-a$ (with $r < a$), and
Γ_3 is the circle $|u| = r$ with positive orientation.

Lemma 11 *The second moment $M(x)$ is given by*

$$M(x) = \frac{1}{2\pi i} \int_\Gamma \mathcal{M}(u) e^{xu} du, \tag{63}$$

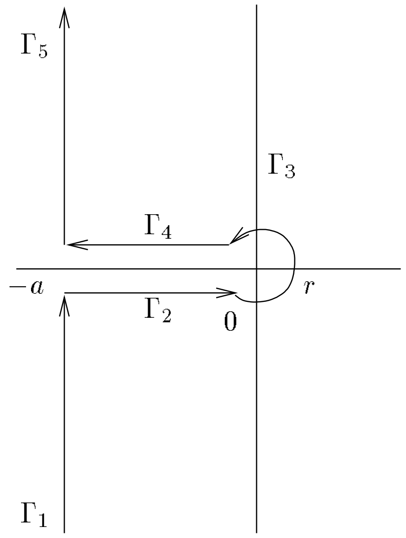

Figure 2: Γ contour of integration.

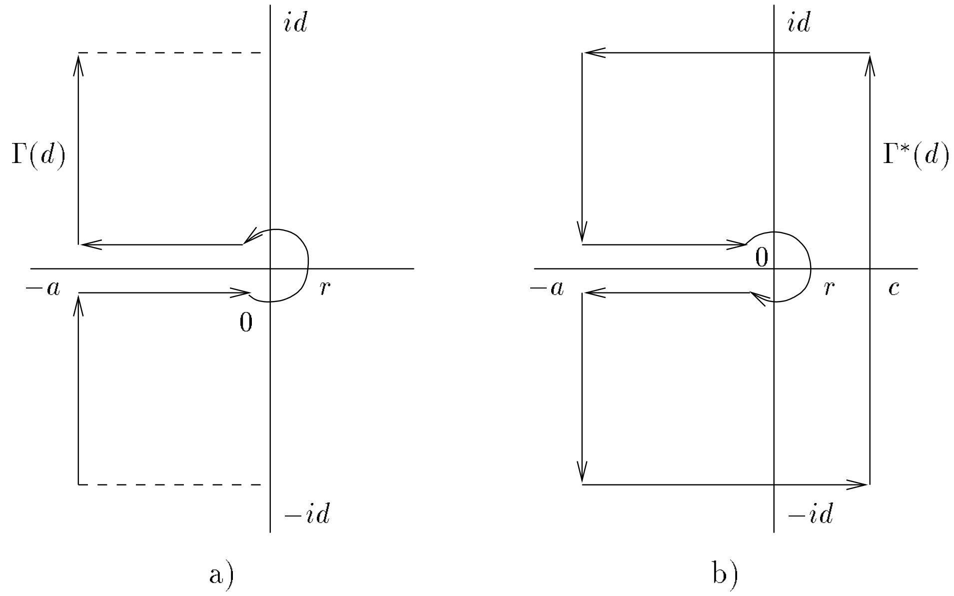

Figure 3: Contours $\Gamma(d)$ and $\Gamma^*(d)$.

Proof: By the inversion formula for Laplace transforms,

$$M(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}(u) e^{xu} du, \tag{64}$$

the integration path being the vertical line $\Re u = c$ (c any positive real), directed upward. We want to show that this gives the same function as (63).

For any $d > 0$, let $\Gamma(d)$ be the portion of Γ contained between the horizontal lines $\Im u = -id$ and $\Im u = +id$; $\Gamma(d)$ is sketched in Fig. 3a.

Let $\Gamma^*(d)$ be the closed contour sketched in Fig. 3b. By Lemma 9, $\mathcal{M}(u)$ is analytic inside $\Gamma^*(d)$ and has a continuous extension to the boundary of $\Gamma^*(d)$. Hence, by Cauchy's theorem,

$$\int_{\Gamma^*(d)} \mathcal{M}(u) e^{xu} du = 0. \tag{65}$$

Define $\Gamma_1^*(d)$ and $\Gamma_2^*(d)$ as the directed horizontal line segments in $\Gamma^*(d)$ passing respectively through $-id$ and id, and rewrite (65) as

$$\int_{\Gamma(d)} \mathcal{M}(u) e^{xu} du = \int_{c-id}^{c+id} \mathcal{M}(u) e^{xu} du + \int_{\Gamma_1^*(d)} \mathcal{M}(u) e^{xu} du + \int_{\Gamma_2^*(d)} \mathcal{M}(u) e^{xu} du. \tag{66}$$

¿From the estimate in Lemma 10, we conclude that

$$\lim_{L\to\infty} \int_{\Gamma_1^*(L)} \mathcal{M}(u) e^{xu} du = \lim_{L\to\infty} \int_{\Gamma_2^*(L)} \mathcal{M}(u) e^{xu} du = 0. \tag{67}$$

Letting $L \to \infty$ in (66) and using (67), we find that

$$\int_{\Gamma} \mathcal{M}(u) e^{xu} du = \int_{c-i\infty}^{c+i\infty} \mathcal{M}(u) e^{xu} du,$$

which is what we set out to prove. ∎

After evaluating (or estimating) various integrals, we will be ready for the proof of Theorem 7; the calculations are given in the next three lemmas.

Lemma 12 *Let $\mathcal{C}$ be the positively oriented circle of radius $r > 0$ centered at the origin. Let $\log u$ denote the principal value of the logarithm and let j be any integer. Then*

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\log u}{u^j} du &= \log r, \quad \text{if } j = 1, \\ &= \frac{(-1)^j}{(j-1) r^{j-1}}, \quad \text{if } j \neq 1. \end{aligned}$$

Proof: Parametrize $\mathcal{C}$ by $u = re^{i\theta}$, $-\pi \leq \theta \leq \pi$. Then $\log u = \log r + i\theta$, so that

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\log u}{u^j} du = \frac{1}{2\pi} \left[\frac{\log r}{r^{j-1}} \int_{-\pi}^{\pi} e^{-i(j-1)\theta} d\theta + \frac{i}{r^{j-1}} \int_{-\pi}^{\pi} \theta e^{-i(j-1)\theta} d\theta \right]. \tag{68}$$

Evaluating the two integrals on the right-hand side, we get

$$\begin{aligned}\int_{-\pi}^{\pi} e^{-i(j-1)\theta} d\theta &= 2\pi, \text{ if } j = 1, \\ &= 0, \text{ if } j \neq 1\end{aligned}$$

and

$$\begin{aligned}\int_{-\pi}^{\pi} \theta e^{i(j-1)\theta} d\theta &= 0, \text{ if } j = 1, \\ &= \frac{2\pi i(-1)^{j-1}}{j-1}, \text{ if } j \neq 1,\end{aligned}$$

which on substitution into (68) proves the lemma. ∎

In the sequel, Lemma 12 is needed only for $j = 1, 2$. In the next lemma, γ denotes Euler's constant.

Lemma 13 *For given $a > 0$, $x > 1$, there exists a function $c_1(a, x) = O(\frac{e^{-ax}}{x})$, $r \to \infty$, such that*

$$\int_r^a \frac{e^{xu}}{u} du = -\log rx - \gamma + c_1(a, x) + o(1), \tag{69}$$

and a function $c_2(a, x) = O(e^{-ax})$, $r \to \infty$, such that

$$\int_r^a \frac{e^{-xu}}{u^2} du = \frac{1}{r} + (\gamma - 1)x + x \log rx + c_2(a, x) + o(1), \tag{70}$$

as $r \to \infty$, where the constants hidden by the $O(.)$ notation depend only on a (not r or x), and the constants hidden by the $o(.)$ notation depend on a and x.

Remark: A formula, not needed here, could have been given with the denominator of the integrand generalized to u^j, j an integer. ∎

Proof: To prove (69), assume, as we may, that $a > r > 0$ and write

$$\begin{aligned}\int_r^a \frac{e^{xu}}{u} du &= \int_{rx}^{ax} \frac{e-u}{u} du \\ &= \int_{rx}^{\infty} \frac{e^{-u}}{u} du - \int_{ax}^{\infty} \frac{e^{-u}}{u} du.\end{aligned} \tag{71}$$

The second integral is bounded by

$$\int_{ax}^{\infty} \frac{e^{-u}}{u} du < \frac{1}{ax} \int_{ax}^{\infty} e^{-u} du = \frac{e^{-ax}}{ax}, \tag{72}$$

and an integration by parts gives

$$\begin{aligned}\int_{rx}^{\infty} \frac{e^{-u}}{u} du &= e^{-rx} \log rx + \int_{rx}^{\infty} \log u \cdot e^{-u} du \\ &= -\log rx + \int_0^{\infty} \log u \cdot e^{-u} du \\ &\quad +(1 - e^{-rx}) \log rx - \int_0^{rx} \log u \cdot e^{-u} du.\end{aligned} \tag{73}$$

Since $-\gamma = \int_0^\infty \log u \cdot e^{-u} du$ [1, p.258], and since the last two terms in (73) tend to 0 as $r \to \infty$, substitution of (72) and (73) into (71) proves (69).

To prove (70), integrate by parts to obtain

$$\begin{aligned}
\int_r^a \frac{e^{-xu}}{u^2} du &= -\frac{e^{-ax}}{a} + \frac{e^{-rx}}{r} - x \int_r^a \frac{e^{-xu}}{u} du \\
&= \frac{e^{-ax}}{a} + \frac{e^{-rx} - 1 + rx}{r} + \frac{1 - rx}{r} - x \int_r^a \frac{e^{-xu}}{u} du,
\end{aligned}$$

so (70) follows from (69) and

$$\lim_{r \to \infty} \frac{e^{-rx} - 1 + rx}{r} = 0.$$

∎

Lemma 14 *Let $f(u)$ be a function analytic on $[0, a]$ for given $a > 0$, and let $f(u) = \sum_{j=0}^\infty q_j u^j$ about $u = 0$. Then*

$$\int_0^a f(u) e^{-xu} du = \sum_{j=0}^J \frac{q_j j!}{x^{j+1}} + O\left(\frac{1}{x^{J+2}}\right), \ x \to \infty, \tag{74}$$

for every integer $J \geq 0$.

Proof: We have

$$\begin{aligned}
\int_0^a u^j e^{-xu} du &= \frac{1}{x^{j+1}} \int_0^{ax} u^j e^{-u} du \\
&= \frac{1}{x^{j+1}} \left[\int_0^\infty u^j e^{-u} du - \int_{ax}^\infty u^j e^{-u} du \right] \\
&= \frac{j!}{x^{j+1}} - \frac{1}{x^{j+1}} \int_{ax}^\infty u^j e^{-u} du. \qquad (75)
\end{aligned}$$

By l'Hospital's rule,

$$\int_x^\infty u^j e^{-u} du \sim x^j e^{-x},$$

as $j \to \infty$, so (75) can be written

$$\int_0^a u^j e^{-xu} du = \sum_{j=0}^J \frac{j!}{x^{j+1}} + O\left(\frac{e^{-x}}{x}\right),$$

as $x \to \infty$. This proves (74), since $f(u) = \sum_{j=0}^J q_j u^j + O(u^{J+1})$ for every $J \geq 0$. ∎

3.4 Proof of Theorem 7

By Theorem 11,

$$M(x) = \frac{1}{2\pi i}\sum_{j=1}^{5}\int_{\Gamma_j}\mathcal{M}(u)e^{xu}du \tag{76}$$

We compute the sum with the terms combined as $\int_{\Gamma_1}+\int_{\Gamma_5}$, $\int_{\Gamma_2}+\int_{\Gamma_4}$, and $\int_{\Gamma_3}$. First, we write

$$\int_{\Gamma_5}\mathcal{M}(u)e^{xu}du = ie^{-ax}\int_0^\infty \mathcal{M}(-a+iy)e^{ixy}dy. \tag{77}$$

The estimate in Lemma 10 shows that

$$\int_0^\infty |\mathcal{M}(-a+iy)|dy < \infty$$

so (77) yields the desired estimate

$$\int_{\Gamma_5}\mathcal{M}(u)e^{xu}du = O(e^{-ax}).$$

A similar argument gives the same bound for the integral over Γ_1, so

$$\left(\int_{\Gamma_1}+\int_{\Gamma_5}\right)\mathcal{M}(u)e^{xu}du = O(e^{-ax}). \tag{78}$$

To compute $\int_{\Gamma_2}+\int_{\Gamma_4}$, we use formula (56) for $\mathcal{M}(u)$. Now $\mathcal{M}_1(u)$ and $\mathcal{M}_2(u)$ take identical values on Γ_2 and Γ_4, while $\log u$ along Γ_4 becomes $\log u + 2\pi i$ along Γ_2. These observations yield

$$\left(\int_{\Gamma_2}+\int_{\Gamma_4}\right)\mathcal{M}(u)e^{xu}du = -2\pi i\int_{-a}^{-r}\mathcal{M}_2(u)e^{xu}du = -2\pi i\int_r^a \mathcal{M}_2(-u)e^{-xu}du. \tag{79}$$

Let $H_2(u) := \mathcal{M}_2(u) - B_{-2}/u^2 - B_{-1}/u$, which by (58) and (59) is entire. Rewrite (79) as

$$\frac{1}{2\pi i}\left(\int_{\Gamma_2}+\int_{\Gamma_4}\right)\mathcal{M}_2(u)e^{xu}du = -B_{-2}\int_r^a\frac{e^{xu}}{u^2}du + B_{-1}\int_r^a\frac{e^{xu}}{u}du - \int_0^a H_2(-u)e^{-xu}du + o(1),$$

which by (69) and (70) becomes

$$\begin{aligned}\frac{1}{2\pi i}\left(\int_{\Gamma_2}+\int_{\Gamma_4}\right)\mathcal{M}(u)e^{xu}du &= -B_{-2}[1/r+(r-1)x+x\log rx] - B_{-1}[\log rx+\gamma]\\ &\quad -\int_0^a H_2(-u)e^{-xu}du + c_2(a,x) + o(1),\end{aligned} \tag{80}$$

where $c_2(a,x) = O(e^{-ax})$.

Finally, by (56), we have for the integral around Γ_3,

$$\int_{\Gamma_3}\mathcal{M}(u)e^{xu}du = \int_{\Gamma_3}\mathcal{M}_1(u)e^{xu}du + \int_{\Gamma_3}\mathcal{M}_2(u)\log u\cdot e^{xu}du. \tag{81}$$

Evaluating $\int_{\Gamma_3} \mathcal{M}_1(u)e^{xu}du$ by the residue theorem and expanding $\mathcal{M}_2(u)e^{xu}$ about $u=0$, (81) becomes

$$\frac{1}{2\pi i}\int_{\Gamma_3} \mathcal{M}(u)e^{xu}du = \left(\frac{A_{-3}}{2}x^2 + A_{-2}x + A_1\right) + \frac{B_{-2}}{2\pi i}\int_{\Gamma_3}\frac{\log u}{u^2}du + \frac{B_{-2}x + B_{-1}}{2\pi i}\int_{\Gamma_3}\frac{\log u}{u}du + o(1),$$

which by Lemma 12 leads to

$$\frac{1}{2\pi i}\int_{\Gamma_3} \mathcal{M}(u)e^{xu}du = \left(\frac{A_{-3}}{2}x^2 + A_{-2}x + A_{-1} + \frac{B{-2}}{r} + (B_{-2}x + B_{-1}\right)\log x + o(1). \quad (82)$$

Finally, apply (74) to (80) and (60) to (82), sum up (78), (80), and (82), and let $r \to \infty$ to obtain Theorem 7. ∎

4 Final Remarks

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Appendix

Proof of (43): In analogy with Section 5, we begin with the following recurrence for the second moment:

$$\begin{aligned}
\mathrm{E}\tilde{N}_{x+1}^2 &= \int_0^x \mathrm{E}[\tilde{N}_x^2|y]\frac{dy}{x} \\
&= \int_0^x \mathrm{E}[1+\tilde{N}_y^2+\tilde{N}_{x-y}^2+2\tilde{N}_y+2\tilde{N}_{x-y}+\tilde{N}_y\tilde{N}_{x-y}]\frac{dy}{x} \\
&= 1+\frac{2}{x}\int_0^x \mathrm{E}\tilde{N}_y^2 dy+\frac{4}{x}\int_0^x \mathrm{E}\tilde{N}_y dy+\frac{2}{x}\int_0^x \mathrm{E}\tilde{N}_y\tilde{N}_{x-y}dy \qquad (83)
\end{aligned}$$

The first-moment transform $\mathcal{F}(u) := \int_0^\infty \mathrm{E}\tilde{N}_x e^{-ux}dx$ is given by (see [16])

$$\mathcal{F}(u) = \frac{e^{-u}\tilde{\alpha}(u)}{u^2\beta(u)}, \qquad (84)$$

where $\beta(.)$ is given by (6) and $\tilde{\alpha}(u) := \alpha_* - \alpha(u)$ with $\alpha(.)$ given by (2). Define the transform of the second moment $\mathcal{G}(u) := \int_0^\infty \mathrm{E}\tilde{N}_x^2 e^{-ux}dx$, take the transform of both sides of (83) and obtain after routine manipulations

$$\frac{d\mathcal{G}}{du} = -\left(1+\frac{2e^{-u}}{u}\right)\mathcal{G} - \frac{e^{-u}}{u}H(u),\ u>0, \qquad (85)$$

where

$$H(u) = 1 + u\mathcal{F}(u) + 2(u\mathcal{F}(u))^2. \qquad (86)$$