Homework Exercises, CSEE E6180

1. Use the Monte Carlo technique to evaluate some standard constant such as π, e, γ (Euler's constant), etc. or some specific value of some classical integral function like the normal distribution, the exponential integral, the sine integral, etc. Submit your program in C along with a few results that illustrate the accuracy of the method (and the random number generator).

2. Use simulation to estimate the error in the normal approximation, in particular, the use of the normal distribution in place of the actual distribution for the sum of k independent random variables, each having the uniform distribution on $[0, 1]$. Take a few values of fairly small k, and for each generate an empirical distribution to compare with the normal.

3. The following simple result is the basis for a Monte Carlo evaluation of e. Fill an array $a_1, \ldots, a_n$ with integers drawn uniformly at random from $\{1, \ldots, n\}$ the samples are taken with replacement and are independent. Verify that the probability that $a_k = k$ holds for at least one $k \in \{1, \ldots, n\}$ tends to $1 - 1/e$ as $n \to \infty$.

4. Prove the memoryless property for the geometric distribution.

5. Let U_1, U_2, U_3 be independent r.v.'s uniformly distributed on $[0, 1]$.

   - Show that $X = \max(U_1, U_2)$ and $Y = U_3^{1/2}$ have the same distribution.
   - What is the distribution of the r.v. $Z = -\ln(U_1 U_2 U_3)$?

6. Let n balls be placed independently and uniformly at random into n initially empty boxes. What is the probability that exactly one box will remain empty at the end of the process? (See Feller, Vol I, for example.)

7. Jobs arrive in a rate-λ Poisson stream at a computer center with two computers called A and B. We split the arriving work evenly between the two computers using one of the following two splitting rules:

Rule 1: Each arriving job flips a fair coin to decide which computer queue to join; let's say heads it goes to computer A and tails it goes to computer B.

Rule 2: Assuming the jobs are numbered in sequence, the even numbered jobs are directed to computer A and the odd numbered jobs are directed to computer B.

Clearly, Rule 1 is randomized and Rule 2 is deterministic. Under a given rule, the arrival processes at the two computers are identical stochastically. Consider one such process. How would you describe it under Rule

1? under Rule 2? Is the process under either rule a Poisson process? under both rules? If the process differs under the two rules, describe the difference. Is there a sense in which one rule is preferable to the other? Explain.

8. Apply Result 1.7 in the notes to derive the stationary probability distribution $\{p_n\}$ for the $M/M/1$ queue and the condition for its existence (the stability condition).

9. Consider a Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ with the transition matrix
$$\begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 3/4 & 1/4 \end{bmatrix}$$
Classify the states as transient, recurrent, absorbing. Is the chain irreducible? aperiodic?. What is its stationary distribution, if any, when started in state 6?

10. Suppose the transition matrix of a Markov chain has nothing but 1's and 0's. Give a detailed account of the ways this chain can behave in terms of the properties of recurrence, transience, absorption, irreducibility, periodicity.

11. For the $M/M/\infty$ queue, assume the system is empty at time 0, and compute the generating function for the state probabilities $P_n(t)$, and from this function conclude that $\{P_n(t)\}_{n \geq 0}$ is a Poisson distribution with rate parameter $\rho(1 - \exp[-\mu t])$, where $\rho = \lambda/\mu$.

12. Prove that the mean $M(t)$ of the distribution $\{P_n(t)\}$ of the $M/M/\infty$ queue satisfies the differential equation
$$M'(t) = \lambda - \mu M(t)$$
Compute this differential equation directly from the differential equations for $P_n(t)$.

Solve this equation assuming the initial state is $k \geq 0$ and point out:

(i) how fast is the approach to statistical equilibrium (i.e., the stationary mean) as $t \to \infty$?

(ii) how fast does the dependence of $M(t)$ on the initial state decay as $t \to \infty$?

13. A scientist starts a job (computation) at 9am on a given day; the job never halts except when the system crashes or when stopped intentionally by the scientist. The status of the job is checked every day at 9am; these are the only times the computation is checked. If at a check the job is running, the scientist stops it, has the current state saved in a secure storage device, and then allows the job to continue. Suppose this takes place in a negligible amount of time.

    If the job is not running when checked, it's because at least one system crash occurred in the previous 24 hours; this occurs with probability p independently from one 24 hour period to the next, each such period beginning at 9am.

    In the case of a crash, the scientist restarts the computation at the state saved the previous day. Again, this takes negligible time. Each time a check is made, and the job is found to be still running, the scientist decides with probability q to terminate the job (e.g., the scientist decides that she has waited long enough for the convergence of some process).

    Problem 1: Find the distribution of the number D of days (including days where computation time was lost due to a crash) before the scientist decides to terminate the job.

    Problem 2: Leave the checking and stochastic crash process as is, but change the Bernoulli process of determining when to terminate the job as follows. For a given N, the scientist knows that after running the program for N crash-free days, she will terminate the job. Give the normal approximation for the quantity D that applies when N is large.

14. Let $H^*(s)$ be the Laplace transform of the busy period of the $M/G/1$ queue, with λ the arrival rate. We discussed in class the approach to proving

    $$H^*(s) = G^*(s + \lambda - \lambda H^*(s))$$

    where $G^*(s)$ is the Laplace transform of the service time distribution. Please work out the details of this derivation, and then find the first moment of the busy period distribution from the above formula.

15. Consider an $M/G/1$ queue with exceptional service for the jobs that begin busy periods. Let $F(x)$ be the service time distribution for the first job of a busy period, and let $G(x)$ be the service time distribution for the remaining jobs. We retain the independence assumptions of the $M/G/1$ queue. Compute the generating function $P(z)$ of the stationary distribution of the number in system in terms of the transforms corresponding to F and G. What can you say about the stability condition? Compute the mean number in system from $P(z)$.