BASIC ELEMENTS OF QUEUEING THEORY
Application to the Modelling of Computer Systems

Lecture Notes*

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Contents

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1 Markov Process

A *Markov process*[1] is a stochastic process $(X(t),\, t \in T)$, $X(t) \in E \subset \mathbb{R}$, such that

$$P(X(t)) \le x \mid X(t_1) = x_1, \ldots, X(t_n) = x_n) = P(X(t) \le x \mid X(t_n) = x_n) \qquad (1)$$

for all $x_1, \ldots, x_n, x \in E$, $t_1, \ldots, t_n, t \in T$ with $t_1 < t_2 < \cdots < t_n < t$.

Intuitively, (1) says that the *probabilistic future* of the process depends only on the *current* state and not upon the *history* of the process. In other words, the entire history of the process is summarized in the current state.

Although this definition applies to Markov processes with continuous state-space, we shall be mostly concerned with discrete-space Markov processes, commonly referred to as *Markov chains*.

We shall distinguish between discrete-time Markov chains and continuous-time Markov chains.

1.1 Discrete-Time Markov Chain

A discrete-time Markov Chain (M.C.) is a discrete-time (with index set $\mathbb{N}$) discrete-space (with state-space $I = \mathbb{N}$ if infinite and $I \subset \mathbb{N}$ if finite) stochastic process $(X_n,\, n \in \mathbb{N})$ such that for all $n \ge 0$

$$P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i) = P(X_{n+1} = j \mid X_n = i) \qquad (2)$$

for all $i_0, \ldots, i_{n-1}, i, j \in I$.

<u>From now on a discrete-time M.C. will simply be referred to as a M.C.</u>

A M.C. is called a *finite-state* M.C. if the set I is finite.

A M.C. is *homogeneous* if $P(X_{n+1} = j \mid X_n = i)$ does not depend on n for all $i, j \in I$. If so, we shall write

$$p_{ij} = P(X_{n+1} = j \mid X_n = i) \quad \forall i, j \in I.$$

p_{ij} is the *one-step transition probability* from state i to state j. Unless otherwise mentioned we shall only consider homogeneous M.C.'s.

[1] A. A. Markov was a Russian mathematician.

Define $P = [p_{ij}]$ to be the transition matrix of a M.C., namely,

$$P = \begin{pmatrix} p_{00} & p_{01} & \dots & p_{0j} & \dots \\ p_{10} & p_{11} & \dots & p_{1j} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i0} & p_{i1} & \dots & p_{ij} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{3}$$

We must have

$$p_{ij} \geq 0 \quad \forall i, j \in I \tag{4}$$

$$\sum_{j \in I} p_{ij} = 1 \quad \forall i \in I. \tag{5}$$

Equation (5) is a consequence of axiom (b) of a probability measure and says that the sum of the elements in each row is 1. A matrix that satisfies (4) and (5) is called a *stochastic matrix*.

If the state-space I is finite (say, with k states) then P is an k-by-k matrix; otherwise P has infinite dimension.

Example 1.1 Consider a sequence of Bernoulli trials in which the probability of success (S) on each trial is p and of failure (F) is q, where $p + q = 1$, $0 < p < 1$. Let the state of the process at trial n (i.e., X_n) be the number of uninterrupted successes that have been completed at this point. For instance, if the first 5 outcomes where SFSSF then $X_0 = 1$, $X_1 = 0$, $X_2 = 1$, $X_3 = 2$ and $X_4 = 0$. The transition matrix is given by

$$P = \begin{pmatrix} q & p & 0 & 0 & 0 & \dots \\ q & 0 & p & 0 & 0 & \dots \\ q & 0 & 0 & p & 0 & \dots \\ q & 0 & 0 & 0 & p & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

The state 0 can be reached in one transition from any state while the state $i + 1$, $i \geq 0$, can only be reached from the state i (with the probability p). Observe that this M.C. is clearly homogeneous.

♡

We now define the n-step transition probabilities $p_{ij}^{(n)}$ by

$$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i) \tag{6}$$

for all $i, j \in I$, $n \geq 0$. $p_{ij}^{(n)}$ is the probability of going from state i to state j in n steps.

Result 1.1 (Chapman-Kolmogorov equation) *For all $n \geq 0$, $m \geq 0$, $i, j \in I$, we have*

$$p_{ij}^{(n+m)} = \sum_{k \in I} p_{ik}^{(n)} \, p_{kj}^{(m)} \tag{7}$$

or, in matrix notation,

$$P^{(n+m)} = P^{(n)} \, P^{(m)} \tag{8}$$

where $P^{(n)} := \left[p_{ij}^{(n)}\right]$. Therefore,

$$P^{(n)} = P^n \quad \forall n \geq 1 \tag{9}$$

where P^n is the n-th power of the matrix P.

□

The Chapman-Kolmogorov equation merely says that if we are to travel from state i to state j in $n+m$ steps then we must do so by first traveling from state i to *some* state k in n steps and then from state k to state j in m more steps.

The proof of Result 1.1 goes as follows. We have

$$\begin{aligned}
p_{ij}^{(n+m)} &= P(X_{n+m} = j \mid X_0 = i) \\
&= \sum_{k \in I} P(X_{n+m} = j, X_n = k \mid X_0 = i) \quad \text{by axiom (b) of a probability measure} \\
&= \sum_{k \in I} P(X_{n+m} = j \mid X_0 = i, X_n = k) \\
&\quad \times P(X_n = k \mid X_0 = i) \text{ by the generalized Bayes' formula} \\
&= \sum_{k \in I} P(X_{n+m} = j \mid X_n = k) \, P(X_n = k \mid X_0 = i) \quad \text{from the Markov property (2)} \\
&= \sum_{k \in I} p_{kj}^{(m)} \, p_{ik}^{(n)}
\end{aligned}$$

since the M.C. is homogeneous, which proves (7) and (8).

Let us now establish (9). Since $P^{(1)} = P$, we see from (8) that $P^{(2)} = P^2$, $P^{(3)} = P^{(2)} \, P = P^3$ and, more generally, that $P^{(n)} = P^n$ for all $n \geq 1$. This concludes the proof.

■

Example 1.2 Consider a communication system that transmits the digits 0 and 1 through several stages. At each stage, the probability that the same digit will be received by the next stage is 0.75. What is the probability that a 0 that is entered at the first stage is received as a 0 at the fifth stage?

We want to find $p_{00}^{(5)}$ for a M.C. with transition matrix P given by

$$P = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}.$$

From Result 1.1 we know that $p_{00}^{(5)}$ is the $(1,1)$-entry of the matrix P^5. We find $p_{0,0}^{(5)} = 0.515625$ (compute P^2, then compute P^4 as the product of P^2 by itself, and finally compute P^5 as the product of matrices P^4 and P).

♡

So far, we have only been dealing with conditional probabilities. For instance, $p_{ij}^{(n)}$ is the probability that the system in state j at time n given it was in state i at time 0. We have shown in Result 1.1 that this probability is given by the (i,j)-entry of the power matrix P^n. What we would like to do now is to compute the unconditional probability that the system is in state j at time n, namely, we would like to compute $\pi_n(i) := P(X_n = i)$.

This quantity can only be computed if we provide the initial d.f. of X_0, that is, if we provide $\pi_0(i) = P(X_0 = i)$ for all $i \in I$, where of course $\sum_{i\in I} \pi_0(i) = 1$.

In that case, we have from Bayes' formula

$$\begin{aligned} P(X_n = j) &= \sum_{i\in I} P(X_n = j \mid X_0 = i)\, \pi_0(i) \\ &= \sum_{i\in I} p_{ij}^{(n)}\, \pi_0(i) \end{aligned}$$

from Result 1.1 or, equivalenty, in matrix notation,

Result 1.2 *For all $n \geq 1$,*

$$\pi_n = \pi_0\, P^n. \tag{10}$$

where $\pi_m := (\pi_m(0), \pi_m(1), \ldots)$ for all $m \geq 0$. From (10) we deduce that (one can also obtain this result directly)

$$\pi_{n+1} = \pi_n\, P \quad \forall n \geq 0. \tag{11}$$

□

Assume that the limiting state distribution function $\lim_{n\to\infty} \pi_n(i)$ exists for all $i \in I$. Call it $\pi(i)$ and let $\pi = (\pi(0), \pi(1), \ldots)$.

How can one compute π? Owing to (10) a natural answer is "by solving the system of linear equations defined by $\pi = \pi\, P$" to which one should add the normalization condition $\sum_{i\in I} \pi(i) = 1$ or, in matrix notation, $\pi\, \mathbf{1} = 1$, where $\mathbf{1}$ is the column vector where every component is 1.

We shall now give conditions under which the above results hold (i.e., $(\pi_n(0), \pi_n(1), \ldots)$ has a limit as n goes to infinity and this limit solves the system of equations $\pi = \pi\, P$ and $\pi\, \mathbf{1} = 1$).

To do so, we need to introduce the notion of *communication* between the states. We shall say that a state j is *reachable* from a state i if $p_{ij}^{(n)} > 0$ for some $n \geq 1$. If j is reachable from i and if i is reachable from j then we say that i and j communicate, and write $i \leftrightarrow j$.

A M.C. is *irreducible* if $i \leftrightarrow j$ for all $i, j \in I$.

For every state $i \in I$, define the integer $d(i)$ as the largest common divisor of all integers n such that $p_{ii}^{(n)} > 0$. If $d(i) = 1$ then the state i is *aperiodic*.

A M.C. chain is *aperiodic* if all states are aperiodic.

We have the following fundamental result of M.C. theory.

Result 1.3 (Invariant measure of a M.C.) *If a M.C. with transition matrix P is irreducible and aperiodic, and if the system of equations*

$$\begin{aligned} \pi &= \pi P \\ \pi\, \mathbf{1} &= 1 \end{aligned}$$

has a strictly positive solution (i.e., for all $i \in I$, $\pi(i)$, the ith element of the row vector π, is strictly positive) then

$$\pi(i) = \lim_{n\to\infty} \pi_n(i) \tag{12}$$

for all $i \in I$, independently of the initial distribution.

□

We shall not prove this result. The equation $\pi = \pi\, P$ is called the *invariant equation* and π is usually referred to as the *invariant measure*.

1.1.1 A Communication Line with Error Transmissions

We consider an infinite stream of packets that arrive at a gateway of a communication network, seeking admittance in the network. For the sake of simplicity we assume that the packets may only arrive in the intervals of time $(n, n+1)$ for all $n \geq 0$ (i.e., we assume that packets do not arrive at times $n = 0, 1, 2, \ldots$). Upon arrival, a packet enters a buffer of infinite dimension. Let $A_n \in \{0, 1\}$ be the number of packets arriving in the interval of time $(n, n+1)$. We assume that for each n, A_n follows a Bernoulli r.v. with parameter a, $0 < a < 1$.

We assume that one unit of time is needed to transmit a packet and that transmissions only start at times $n = 0, 1, 2, \ldots$ provided that the buffer is nonempty (it does not matter which packet is transmitted).

We assume that transmission errors may occur. More precisely, a packet transmitted in any interval of time $[n, n+1)$ is transmitted in error with the probability $0 \leq 1 - p < 1$. When this happens, the packet is retransmitted in the next time-slot and the procedure is repeated until the transmission is a success eventually (which occurs with probability one since $1 - p < 1$).

We assume that the r.v.'s $(A_n)_n$ are mutually independent r.v.'s, that transmission errors are mutually independent, and further independent of the r.v.s' $(A_n)_n$.

Let X_n be the number of packets in the buffer at time n. Our objective is to compute $\pi(i) := \lim_{n\to\infty} P(X_n = i)$ for all $i \in \mathbb{N}$ when this limit exists.

We have the following *evolution equation* for this system:

$$
\begin{aligned}
X_{n+1} &= A_n \quad \text{if } X_n = 0 && (13)\\
X_{n+1} &= X_n + A_n - D_n \quad \text{if } X_n > 0 && (14)
\end{aligned}
$$

for all $n \in \mathbb{N}$, where $D_n \in \{0, 1\}$ gives the number of packet transmitted with success in the interval of time $[n, n+1)$.

Since the r.v.'s A_n and D_n are independent of the r.v.'s $(A_i, D_i, i = 0, 1, \ldots, n-1)$, and therefore of X_n from (13)-(14), it should be clear from (13)-(14) that $(X_n, n \geq 0)$ is a M.C.

We shall however prove this result explicitely.

We have for $i = 0$ and for arbitrary $i_0, \ldots, i_{n-1}, j \in \mathbb{N}$

$$
\begin{aligned}
&P(X_{n+1} = j \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i)\\
&= P(A_n = j \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i) \quad \text{from (13)}
\end{aligned}
$$

$$= \quad P(A_n = j)$$

from the independence assumptions. Hence, for $i = 0$, $P(X_{n+1} = j \,|\, X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i)$ is equal to 0 if $j \geq 2$, is equal to a if $j = 1$, and is equal to $1 - a$ if $j = 0$. This shows that $P(X_{n+1} = j \,|\, X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i)$ is *not* a function of $i_0, \ldots, i_{n-1}$.

Consider now the case when $i \geq 1$. We have

$$\begin{aligned}
&P(X_{n+1} = j \,|\, X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i) \\
&\quad = \quad P(A_n - D_n = j - i \,|\, X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i) \quad \text{from (14)} \\
&\quad = \quad P(A_n - D_n = j - i) \qquad (15)
\end{aligned}$$

from the independence assumptions.

Clearly, $P(A_n - D_n = j - i)$ in (15) is equal to 0 when $|j - i| \geq 2$. On the other hand, routine applications of Bayes formula together with the independence of A_n and D_n yield $P(A_n - D_n = j - i)$ is equal to $(1 - a)p$ when $j = i - 1$, is equal to $ap + (1 - a)(1 - p)$ when $i = j$, and is equal to $(1 - p)a$ when $j = i + 1$. Again, one observes that $P(X_{n+1} = j \,|\, X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i)$ does not depend on $i_0, \ldots, i_{n-1}$ when $i \geq 1$. This proves that $(X_n, n \geq 0)$ is a M.C.

From the above we see that the transition matrix $P = [p_{ij}]$ of this M.C. is given by:

$$\begin{aligned}
p_{00} &= 1 - a \\
p_{01} &= a \\
p_{0j} &= 0 \quad \forall j \geq 2 \\
p_{ii-1} &= (1 - a)p \quad \forall i \geq 1 \\
p_{ii} &= ap + (1 - a)(1 - p) \quad \forall i \geq 1 \\
p_{ii+1} &= a(1 - p) \quad \forall i \geq 1 \\
p_{ij} &= 0 \quad \forall i \geq 1,\, j \notin \{i - 1, i, i + 1\}.
\end{aligned}$$

Let us check that this M.C. is irreducible and aperiodic.

Since $p_{ii} > 0$ for all $i \in \mathbb{N}$ we see that this M.C. is aperiodic.

Let $i, j \in \mathbb{N}$ be two arbitrary states. We first show that j is reachable from i. We know that this is true if $i = j$ since $p_{ii} > 0$. If $j > i$ then clearly $p_{ij}^{(j-i)} \geq a^{j-i}\,(1 - p)^{j-i} > 0$ if $i > 0$ and $p_{ij}^{(j-i)} \geq a^j\,(1 - p)^{j-1} > 0$ if $i = 0$; if $0 \leq j < i$ then clearly $p_{ij}^{(i-j)} \geq (1 - a)^{i-j}\,p^{i-j} > 0$. Hence, j is reachable from i. Since i and j can be interchanged, we have in fact established that $i \leftrightarrow j$. This shows that the M.C. is irreducible since i and j are arbitrary states.

We may therefore apply Result 1.3. This result says that we must find a strictly positive solution $\pi = (\pi(0), \pi(1), \ldots)$ to the system of equations

$$\begin{aligned}
\pi(0) &= \pi(0)\,(1-a) + \pi(1)\,(1-a)p && (16)\\
\pi(1) &= \pi(0)\,a + \pi(1)\,(ap + (1-a)(1-p)) + \pi(2)\,(1-a)p && (17)\\
\pi(j) &= \pi(j-1)\,a(1-p) + \pi(j)\,(ap + (1-a)(1-p)) + \pi(j+1)(1-a)p,\ \forall j \geq 2 && (18)
\end{aligned}$$

such that

$$\sum_{j=0}^{\infty} \pi(j) = 1. \tag{19}$$

It is easily seen that

$$\begin{aligned}
\pi(1) &= \frac{a}{(1-a)\,p}\,\pi(0) && (20)\\
\pi(j) &= \frac{1}{1-p}\left(\frac{a\,(1-p)}{p\,(1-a)}\right)^{j} \pi(0) \quad \forall j \geq 2 && (21)
\end{aligned}$$

satisfies the system of equations (16)-(18) (hint: determine first $\pi(1)$ and $\pi(2)$ as functions of $\pi(0)$ by using equations (16) and (17), then use equation (18) to recursively determine $\pi(j)$ for $j \geq 3$).

We must now compute $\pi(0)$ such that (19) hold. Introducing the values of $\pi(j)$ obtained above in equation (19) gives after trivial algebra

$$\pi(0)\left(1 + \frac{a}{p\,(1-a)} \sum_{j=0}^{\infty} \left(\frac{a\,(1-p)}{p\,(1-a)}\right)^{j}\right) = 1. \tag{22}$$

Fix $r \geq 0$. Recall that the power series $\sum_{j=0}^{\infty} r^j$ converges if and only if $r < 1$. If $r < 1$ then $\sum_{j=0}^{\infty} r^j = 1/(1-r)$.

Therefore, we see that the factor of $\pi(0)$ in (22) is finite if and only if $a\,(1-p)/p\,(1-a) < 1$, or equivalently, if and only if $a < p$. If $a < p$ then

$$\pi(0) = \frac{p-a}{p}. \tag{23}$$

In summary, we have found a strictly positive solution to the system of equations (16)-(19) if $a < p$. We may therefore conclude from Result 1.3 that $\lim_{n\infty} P(X_n = i)$ exists for all $i \in \mathbb{N}$ if $a < p$, is independent of the initial state, and is given by $\pi(i)$ for all $i \in \mathbb{N}$.

The condition $a < p$ is not surprising, since it simply says that the system is stable if the mean number of arrivals in any interval of time $(n, n+1)$ is strictly smaller than the probability of having a successful transmission in this interval.

We will assume from now on that $a < p$. When the system is stable, clearly the input rate a must be equal to the throughput. Let us check this intuitive result.

Since a packet may leave the system (with probability p) only when the queue in not empty (which occurs with probability $1 - \pi(0)$), the throughput T is given by

$$\begin{aligned} T &= p\,(1 - \pi(0)) \\ &= a \end{aligned}$$

from (23), which is the expected result.

Let Q be the number of packets in the waiting room in steady-state, including the packet being transmitted, if any. We now want to do some flow control. More precisely, we want to determine the input rate a such that $P(Q > k) < \beta$ where $k \in \mathbb{N}$ and $\beta \in (0,1)$ are arbitrary numbers.

Let us compute $P(Q \leq k)$. We have

$$P(Q \leq k) = \sum_{j=0}^{k} \pi(j).$$

After elementary algebra we finally obtain [2]

$$P(Q \leq k) = 1 - \frac{a}{p} \left(\frac{a\,(1-p)}{p\,(1-a)} \right)^k.$$

(Observe from the above result that $\lim_{k\to\infty} P(Q \leq k) = 1$ under the stability condition $a < p$, which simply says that the number of packets in the waiting room is finite with probability one in steady-state.)

In conclusion, if we want $P(Q > k) < \beta$, we must choose a such that $0 \leq a < p$ and

$$\frac{a}{p} \left(\frac{a\,(1-p)}{p\,(1-a)} \right)^k < \beta.$$

Such a result is useful, for instance, for dimensioning the size of the buffer so that the probability of loosing a packet is below a given threshold.

Other interesting performance measures for this system are $E[Q]$, $\mathrm{Var}(Q)$, $P(X_n = j)$ given the d.f. of X_0 is known (use Result 1.2), etc.

[2] Here we use the well-known identity $\sum_{j=0}^{k} r^j = (1 - r^{k+1})/(1-r)$ for all $k \geq 0$ if $0 \leq r < 1$.

1.2 Continuous-Time Markov Chain

A continuous-time Markov chain (denoted as C-M.C.) is a continuous-time (with index set $[0,\infty)$), discrete-space (with state-space I) stochastic process $(X(t),\, t \geq 0)$ such that

$$P(X(t) = j \,|\, X(s_1) = i_1, \ldots, X(s_{n-1}) = i_{n-1}, X(s) = i) = P(X(t) = j \,|\, X(s) = i) \qquad (24)$$

for all $i_1, \ldots, i_{n-1}, i, j \in I$, $0 \leq s_1 < \ldots < s_{n-1} < s < t$.

A C-M.C. is homogenous if

$$P(X(t+u) = j \,|\, X(s+u) = i) = P(X(t) = j \,|\, X(s) = i) := p_{ij}(t-s)$$

for all $i, j \in I$, $0 \leq s < t$, $u \geq 0$.

From now on we shall only consider homogeneous C-M.C.'s.

We have the following analog to Result 1.1:

Result 1.4 (Chapman-Kolmogorov equation for C-M.C.'s) *For all $t > 0$, $s > 0$, $i, j \in I$,*

$$p_{ij}(t+s) = \sum_{k \in I} p_{ik}(t)\, p_{kj}(s). \qquad (25)$$

□

The proof is the same as the proof of 1.1 and is therefore omitted.

Define

$$q_{ii} := \lim_{h \to 0} \frac{p_{ii}(h) - 1}{h} \leq 0 \qquad (26)$$

$$q_{ij} := \lim_{h \to 0} \frac{p_{ij}(h)}{h} \geq 0 \qquad (27)$$

and let Q be the matrix $Q = [q_{ij}]$ (we will assume that these limits always exist. They will exist in all cases to be considered in this course).

The matrix Q is called the *infinitesimal generator* of the C-M.C.. If $I = \mathbb{N}$, then

$$Q = \begin{pmatrix} -\sum_{j \neq 0} q_{0j} & q_{01} & q_{02} & \cdots & \cdots & \cdots & \cdots \\ q_{10} & -\sum_{j \neq 1} q_{1j} & q_{12} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{i0} & \cdots & \cdots & q_{ii-1} & \sum_{j \neq i} q_{ij} & q_{ii+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

In contrast with (5) note that

$$\sum_{j \in I} q_{ij} = 0 \quad \forall i \in I. \tag{28}$$

The quantity q_{ij} has the following interpretation: when the system is in state i then the rate at which it departs state i is q_{ii}, and the rate at which it moves from state i to state j , $j \neq i$, is q_{ij}.

Define the row vector $\pi(t)$ where its i-th component is $P(X(t) = i)$ for all $i \in I$, $t \geq 0$.

One can show that

$$\pi(t) = \pi(0)\, e^{Qt}$$

for all $t \geq 0$. However, this result is not very useful in practice unless we are interested in specific values of t.

In the rest of this section we shall only be concerned with the computation of $\lim_{t\to\infty} P(X(t) = i)$ for all $i \in I$, when these limits exist.

In direct analogy with the definition given for a M.C., we shall say that an homogeneous C-M.C. is *irreducible* if for every state $i \in I$ there exists $s > 0$ such that $p_{ii}(s) > 0$.

Result 1.5 (Limiting d.f. of a C-M.C.) *If a C-M.C. with infinitesimal generator Q is irreducible, and if the system of equations*

$$\begin{aligned} \pi Q &= 0 \\ \pi \mathbf{1} &= 1 \end{aligned}$$

has a strictly positive solution (i.e., for all $i \in I$, $\pi(i)$, the i-th component of the row vector π, is strictly positive) then

$$\pi(i) = \lim_{t\to\infty} P(X(t) = i) \tag{29}$$

for all $i \in I$, independently of the initial distribution.

□

We shall not prove this result. We may compare this result with our earlier equation for discrete-time M.C.'s, namely, $\pi P = \pi$; here P was the transition matrix, whereas the infinitesimal generator Q is a matrix of *transition rates*.

The equation $\pi Q = 0$ in Result 1.5 can be rewritten as

$$\sum_{j \in I} \pi(j)\, q_{ji} = 0$$

for all $i \in I$, or equivalently, cf. (28),

$$\left(\sum_{j \neq i} q_{ij} \right) \pi(i) = \sum_{j \neq i} \pi(j)\, q_{ji}. \tag{30}$$

Like in the case of a birth and death process, equation (30) says that, at equilibrium,

the probability flow out of a state = the probability flow into that state.

Equations (30) are called the balance equations.

Many examples of continuous-time Markov chains will be discussed in the forthcoming lectures.

1.3 Birth and Death Process

A *birth and death process* $(X(t),\, t \geq 0)$ is a continuous-time discrete-space (with state-space $\mathbb{N}$) Markov process such that

(a) $P(X(t+h) = n+1 \,|\, X(t) = n) = \lambda_n h + o(h)$ for $n \geq 0$

(b) $P(X(t+h) = n-1 \,|\, X(t) = n) = \mu_n h + o(h)$ for $n \geq 1$

(c) $P(X(t+h) = n \,|\, X(t) = n) = 1 - (\lambda_n + \mu_n)h + o(h)$, for $n \geq 0$.

The r.v. $X(t)$ may be interpreted as the *size of the population at time* t. In that case, $\lambda_n \geq 0$ gives the birth-rate when the size of the population is n and $\mu_n \geq 0$ gives the death-rate when the size of the population is n with $n \geq 1$. We assume that $\mu_0 = 0$.

What are $P_n(t) = P(X(t) = n)$ for $n \in \mathbb{N}$, $t \geq 0$?

We have:

$$\begin{aligned} P_n(t+h) &= \sum_{k=0}^{\infty} P(X(t+h) = n \,|\, X(t) = k)\, P(X(t) = k) \quad \text{(Bayes' formula)} \\ &= (\lambda_{n-1} h + o(h))\, P_{n-1}(t) + \ (1 - (\lambda_n + \mu_n)h + o(h))\, P_n(t) \\ &\quad + (\mu_{n+1} h + o(h)) P_{n+1}(t) + o(h) \end{aligned} \tag{31}$$

where by convention $\lambda_{-1} = 0$.

Similarly to the proof of Result C.1 we get from (31) the ordinary differential equation

$$\frac{dP_n(t)}{dt} = \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t) \tag{32}$$

for $n \in \mathbb{N}$.

It is a difficult task to solve this differential equation unless we have some conditions on the parameters λ_n and μ_n

An interesting question is the following: what happens when $t \to \infty$? In other words, we are now interested in the *equilibrium* behavior, if it exists. Any "reasonable" system is expected to reach "equilibrium". When the system is in equilibrium (we also say in *steady-state* or *stationary*) then its state does not depend on t.

Assume that $\lim_{t\to\infty} P_n(t) = p_n$ exists for each $n \in \mathbb{N}$. Letting $t \to \infty$ in (32) gives for each $n \in \mathbb{N}$

$$0 = \lambda_{n-1}\, p_{n-1} - (\lambda_n + \mu_n)\, p_n + \mu_{n+1}\, p_{n+1}$$

or, equivalently,

Result 1.6 (Balance equations of a birth and death process)

$$\begin{aligned} \lambda_0\, p_0 &= \mu_1\, p_1 && (33) \\ (\lambda_n + \mu_n)\, p_n &= \lambda_{n-1}\, p_{n-1} + \mu_{n+1}\, p_{n+1} \quad n = 1, 2, \ldots. && (34) \end{aligned}$$

□

Equations (33)-(34) are called the *equilibrium equations* or the *balance equations* of a birth and death process. They have a natural and useful interpretation.

They say that, at equilibrium,

<u>the probability flow out of a state = the probability flow in that state</u>.

This key observation will allow us in most cases to generate the (correct!) equilibrium equations of a system without going through the burden of writing down equations like equation (31).

From Result 1.6 we have:

Result 1.7 *Assume that the series*

$$C := 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots + \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} + \cdots \tag{35}$$

converges (i.e., $C < \infty$). Then, for each $n = 1, 2 \ldots$,

$$p_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\, p_0 \tag{36}$$

where $p_0 = 1/C$.

□

This result is obtained by direct substitution of (36) into (33)-(34). The computation of p_0 relies on the fact that $\sum_{n=0}^{\infty} p_n = 1$.

The condition (35) is called the *stability condition* of a birth and death process.

2 Queueing Theory

Queues are common in computer systems. Thus, there are queues of inquiries waiting to be processed by an interactive computer system, queue of data base requests, queues of I/O requests, etc.

Typically a queue (or queueing system) has one service facility, although there may be more than one server in the service facility, and a waiting room (or buffer) of finite or infinite capacity.

Customers from a population or *source* enter a queueing system to receive some service. Here the word customer is used in its generic sense, and thus maybe a packet in a communication network, a job or a program in a computer system, a request or an inquiry in a database system, etc.

Upon arrival a customer joins the waiting room if all servers in the service center are busy. When a customer has been served, he leaves the queueing system.

A special notation, called Kendall's notation, is used to describe a queueing system. The notation has the form

A/B/c/K

where

- A describes the *interarrival time* distribution
- B the *service time* distribution
- c the number of servers
- K the size of the system capacity (including the servers).

The symbols traditionally used for A and B are

- M for exponential distribution (M stands for Markov)
- D for deterministic distribution
- G (or GI) for general distribution.

When the system capacity is infinite ($K = \infty$) one simply uses the symbol $A/B/c$.

For instance, $M/M/1$, $M/M/c$, $M/G/1$ and $G/M/1$ are very common queueing systems.

2.1 The M/M/1 Queue

In this queueing system the customers arrive according to a Poisson process with rate λ. The time it takes to serve every customer is an exponential r.v. with parameter μ. We say that the customers have exponential service times. The service times are supposed to be mutually independent and further independent of the interarrival times.

When a customer enters an empty system his service starts at once; if the system is nonempty the incoming customer joins the queue. When a service completion occurs, a customer from the queue (we do not need to specify which one for the time being), if any, enters the service facility at once to get served.

Let $X(t)$ be the number of customers in the system at time t.

Result 2.1 *The process $(X(t),\, t \geq 0)$ is a birth and death process with birth rate $\lambda_i = \lambda$ for all $i \geq 0$ and with death rate $\mu_i = \mu$ for all $i \geq 1$.*

$\square$

Proof. Because of the exponential distribution of the interarrival times and of the service times, it should be clear that $(X(t),\, t \geq 0)$ is a Markov process. On the other hand, since the probability of having two events (departures, arrivals) in the interval of time $(t, t+h)$ is $o(h)$ we have

$$
\begin{aligned}
P(X(t+h) = i+1 \,|\, X(t) = i) &= \lambda\, h + o(h) \quad \forall i \geq 0 \\
P(X(t+h) = i-1 \,|\, X(t) = i) &= \mu\, h + o(h) \quad \forall i \geq 1 \\
P(X(t+h) = i \,|\, X(t) = i) &= 1 - (\lambda + \mu)\, h + o(h) \quad \forall i \geq 1 \\
P(X(t+h) = i \,|\, X(t) = i) &= 1 - \lambda\, h + o(h) \quad \text{for } i = 0 \\
P(X(t+h) = j \,|\, X(t) = i) &= o(h) \quad \text{for } |j - i| \geq 2.
\end{aligned} \tag{37}
$$

This shows that $(X(t),\, t \geq 0)$ is a birth and death process.

■

Let $\pi(i)$, $i \geq 0$, be the d.f. of the number of customers in the system in steady-state.

The balance equations for this birth and death process read

$$
\begin{aligned}
\lambda\, \pi(0) &= \mu\, \pi(1) \\
(\lambda + \mu)\, \pi(i) &= \lambda\, \pi(i-1) + \mu\, \pi(i+1) \quad \forall i \geq 1.
\end{aligned}
$$

Define [3]

$$\rho = \frac{\lambda}{\mu}. \tag{38}$$

The quantity ρ is referred to as the *traffic intensity* since it gives the mean quantity of work brought to the system per unit of time.

A direct application of Result 1.7 yields:

Result 2.2 (Stationary queue-length d.f. of an M/M/1 queue) *If* $\rho < 1$ *then*

$$\pi(i) = (1 - \rho)\, \rho^i \tag{39}$$

for all $i \geq 0$.

□

Therefore, the *stability condition* $\rho < 1$ simply says that the system is stable if the work that is brought to the system per unit of time is strictly smaller than the processing rate (which is 1 here since there is only one server).

[3]From now on ρ will always be defined as λ/μ unless otherwise mentioned.

Result 2.2 therefore says that the d.f. of the queue-length in steady-state is a *geometric distribution*.

From (39) we can compute (in particular) the mean number of customers $E[X]$ (still in steady-state). We find

$$E[X] = \frac{\rho}{1-\rho}. \tag{40}$$

Observe that $E[X] \to \infty$ when $\rho \to 1$, so that, in pratice if the system is not stable, then the queue will explode. It is also worth observing that the queue will empty infinitely many times when the system is stable since $\pi(0) = 1 - \rho > 0$.

We may also be interested in the probability that the queue exceeds, say, K customers, in steady-state. From (39) we have

$$P(X \geq K) = \rho^K. \tag{41}$$

What is the throughput T of an M/M/1 in equilibrium? The answer should be $T = \lambda$. Let us check this guess.

We have

$$T = (1 - \pi(0))\, \mu.$$

Since $\pi(0) = 1 - \rho$ from (39) we see that $T = \lambda$ by definition of ρ.

2.2 The M/M/1/K Queue

In practice, queues are always finite. In that case, a new customer is lost when he finds the system full (e.g., telephone calls).

The $M/M/1/K$ may accomodate at most K customers, including the customer in the service facility, if any. Let λ and μ be the rate of the Poisson process for the arrivals and the parameter of the exponential distribution for the service times, respectively.

Let $\pi(i)$, $i = 0, 1, \ldots, K$, be the d.f. of the queue-length in steady-state. The balance equations for this birth and death process read

$$\begin{aligned}
\lambda\, \pi(0) &= \mu\, \pi(1) \\
(\lambda + \mu)\, \pi(i) &= \lambda\, \pi(i-1) + \mu\, \pi(i+1) \qquad \text{for } i = 1, 2 \ldots, K-1 \\
\lambda\, \pi(K-1) &= \mu\, \pi(K).
\end{aligned}$$

Result 2.3 (Stationary queue-length d.f. in an M/M/1/K queue) *If $\rho \neq 1$ then*

$$\pi(i) = \frac{(1-\rho)\,\rho^i}{1-\rho^{K+1}} \tag{42}$$

for $i = 0, 1, \ldots, K$, and $\pi(i) = 0$ for $i > K$.

If $\rho = 1$ then

$$\pi(i) = 1/(K+1) \tag{43}$$

for $i = 0, 1, \ldots, K$, and $\pi(i) = 0$ for $i > K$.

$\square$

Here again the proof of Result 2.3 relies on the fact that $(X(t),\, t \geq 0)$, where $X(t)$ is the number of customers in the system at time t, can be modeled as a birth and death process with birth rates $\lambda_i = \lambda$ for $i = 0, 1, \ldots, K-1$ and $\lambda_i = 0$ for $i \geq K$.

In particular, the probability that an incoming customer is rejected is $\pi(K)$.

2.3 The M/M/c Queue

There are $c \geq 1$ servers and the waiting room has infinite capacity. If more than one server is available when a new customer arrives (which necessarily implies that the waiting room is empty) then the incoming customer may enter any of the free servers.

Let λ and μ be the rate of the Poisson process for the arrivals and the parameter of the exponential distribution for the service times, respectively.

Here again the process $(X(t),\, t \geq 0)$ of the number of customers in the system can be modeled as a birth and death process. The birth rate is $\lambda_i = \lambda$ when $i \geq 0$. The death rate is given by

$$\begin{aligned} \mu_i &= i\mu \quad \text{for } i = 1, 2, \ldots, c-1 \\ &= c\mu \quad \text{for } i \geq c \end{aligned}$$

which can be also written as $\mu_i = \mu \, \min(i, c)$ for all $i \geq 1$.

Using these values of λ_i and μ_i in Result 1.7 yields

Result 2.4 (Stationary queue-length d.f. in an M/M/c queue) *If $\rho < c$ then*

$$\pi(i) = \begin{cases} \pi(0)\,\dfrac{\rho^i}{i!} & \textit{if } i = 0, 1, \ldots, c \\ \\ \pi(0)\,\dfrac{\rho^i\, c^{c-i}}{c!} & \textit{if } i \geq c \end{cases} \tag{44}$$

where

$$\pi(0) = \left[\sum_{i=0}^{c-1} \frac{\rho^i}{i!} + \left(\frac{\rho^c}{c!}\right) \left(\frac{1}{1-\rho/c}\right) \right]^{-1}. \tag{45}$$

□

The probability that an arriving customer is forced to join the queue is given by

$$\begin{aligned} P(\text{queueing}) &= \sum_{i=c}^{\infty} \pi(i) \\ &= \sum_{i=c}^{\infty} \pi(0)\,\frac{\rho^i\, c^{c-i}}{c!}. \end{aligned}$$

Thus,

$$P(\text{queueing}) = \frac{\left(\dfrac{\rho^c}{c!}\right)\left(\dfrac{1}{1-\rho/c}\right)}{\displaystyle\sum_{i=0}^{c-1} \frac{\rho^i}{i!} + \left(\frac{\rho^c}{c!}\right)\left(\frac{1}{1-\rho/c}\right)}. \tag{46}$$

This probability is of wide use in telephony and gives the probability that no trunk (i.e., server) is available for an arriving call (i.e., customer) in a system of c trunks. It is referred to as *Erlang's C formula*.

2.4 The M/M/c/c Queue

Here we have a situation when there are $c \geq 1$ available servers but no waiting room. This is a pure *loss queueing system*. Each newly arriving customer is given its private server; however, if a customer arrives when all the servers are occupied, that customer is lost. Parameters λ and μ are defined as in the previous sections.

The number of busy servers can be modeled as a birth and death process with birth rate

$$\lambda_i = \begin{cases} \lambda & \text{if } i = 0, 1, \ldots, c-1 \\ \\ 0 & \text{if } i \geq c \end{cases}$$

and death rate $\mu_i = i\mu$ for $i = 1, 2, \ldots, c$.

We are interested in determining the limiting d.f. $\pi(i)$ $(i = 0, 1, \ldots, c)$ of the number of busy servers.

Result 2.5 (Stationary server occupation in a M/M/c/c queue)

$$\pi(i) = \pi(0)\,\frac{\rho^i}{i!} \tag{47}$$

for $i = 0, 1, \ldots, c$, $\pi(i) = 0$ *for* $i > c$, *where*

$$\pi(0) = \left[\sum_{i=0}^{c} \frac{\rho^i}{i!}\right]^{-1}. \tag{48}$$

□

This system is also of great interest in telephony. In particular, $\pi(c)$ gives the probability that all trunks (i.e., servers) are busy, and it is given by

$$\pi(c) = \frac{\rho^i/i!}{\sum_{j=0}^{c} \rho^j/j!}. \tag{49}$$

This is the celebrated *Erlang's loss formula* (derived by A. K. Erlang in 1917).

Remarkably enough Result 2.5 is valid for *any service time distribution* and not only for exponential service times! Such a property is called an insensitivity property.

Later on, we will see an extremely useful extension of this model to (in particular) several classes of customers, that has nice applications in the modeling and performance evaluation of multimedia networks.

2.5 The Repairman Model

It is one of the most useful models. There are K machines and a single repairman. Each machine breaks down after a time that is exponentially distributed with parameter α. In other words, α is the rate at which a machine breaks.

When a breadown occurs, a request is sent to the repairman for fixing it. Requests are buffered. It takes an exponentially distributed amount of time with parameter μ for the repairman to repair a machine. In other words, μ is the repair rate.

We assume that "lifetimes" and repair times are all mutually independent.

What is the probability $\pi(i)$ that i machines are up (i.e., working properly)? What is the overall failure rate?

Let $X(t)$ be the number of machines up at time t. It is easily seen that $(X(t),\, t \geq 0)$ is a birth and death process with birth and death rates given by $\lambda_n = \mu$ for $n = 0, 1, \ldots, K-1$, $\lambda_n = 0$ for $n \geq K$ and $\mu_n = n\alpha$ for $n = 1, 2, \ldots, K$, respectively.

We notice that $(X(t),\, t \geq 0)$ has the same behavior as the queue-length process of an $M/M/K/K$ queue!

Hence, by (47) and (48) we find that

$$\pi(i) = \frac{(\mu/\alpha)^i/i!}{C(K, \mu/\alpha)}$$

for $i = 0, 1, \ldots, K$, where

$$C(K, a) := \sum_{i=0}^{K} \frac{a^i}{i!}. \tag{50}$$

The overall failure rate λ_b is given by

$$\begin{aligned}
\lambda_b &= \sum_{i=1}^{K} (\alpha\, i)\, \pi(i) \\
&= \alpha\, \frac{\sum_{i=1}^{K} i\, (\mu/\alpha)^i/i!}{C(K, \mu/\alpha)} \\
&= \mu\, \frac{\sum_{i=0}^{K-1} (\mu/\alpha)^i/i!}{C(K, \mu/\alpha)} \\
&= \mu\, \frac{C(K-1, \mu/\alpha)}{C(K, \mu/\alpha)}.
\end{aligned}$$

Observe that $\pi(0) = 1/C(K, \mu/\alpha)$. Hence, the mean number n_r of machines repaired by unit of time is

$$\begin{aligned}
n_r &= \mu\, (1 - \pi(K)) \\
&= \mu \left(1 - \frac{(\mu/\alpha)^K/K!}{C(K, \mu/\alpha)}\right). \\
&= \mu \left(\frac{C(K, \mu/\alpha) - (\mu/\alpha)^K/K!}{C(K, \mu/\alpha)}\right)
\end{aligned}$$

$$= \mu \frac{C(K-1,\mu/\alpha)}{C(K,\mu/\alpha)}.$$

2.6 Little's formula

So far we have only obtained results for the buffer occupation namely, the limiting distribution of the queue-length, the mean number of customers, etc. These performance measures are of particular interest for a system's manager. What we would like to do now is to address performance issues from a user's perspective, such as, for instance, response times and waiting times.

For this, we need to introduce the most used formula in performance evaluation.

Result 2.6 (Little's formula) *Let $\lambda > 0$ be the arrival rate of customers to a queueing system in steady-state. Let $\overline{N}$ be the mean number of customers in the system and let $\overline{T}$ be the mean sojourn time of a customer (i.e., the sum of its waiting time and of its service time).*

Then,

$$\overline{N} = \lambda \overline{T}. \tag{51}$$

□

This formula is of great interest since very often one knows $\overline{N}$ and λ. It states that the average number of customers in a queueing system in steady-state is equal to the arrival rate of customers to that system, times the average time spent in that system. This result does not make any specific assumption regarding the arrival distribution or the service time distribution; nor does it depend upon the number of servers in the system or upon the particular queueing discipline within the system.

This result has an intuitive explanation: $\overline{N}/\overline{T}$ is the departure rate, which has to be equal to the input rate λ since the system is in steady-state.

Example 2.1 Consider an M/M/1 queue with arrival rate λ and service rate μ. Let $\overline{T}$ (resp. $\overline{W}$) be the mean customer sojourn time, also referred to as the mean customer response time (resp. waiting time).

If $\rho := \lambda/\mu < 1$ (i.e., if the queue is stable) then we know that the mean number of customers N is given by $\overline{N} = \rho/(1-\rho)$ (see 40).

Therefore, by Little's formula,

$$\overline{T} = \frac{\rho}{\lambda\,(1-\rho)} = \frac{1}{\mu\,(1-\rho)} \tag{52}$$

$$\overline{W} = \overline{T} - \frac{1}{\mu} = \frac{\rho}{\mu\,(1-\rho)}\,. \tag{53}$$

Observe that both $\overline{T} \to \infty$ and $\overline{W} \to \infty$ when $\rho \to 1$.

♡

We now give a proof of Little's formula in the case where the system empties infinitely often.

Starting from an empty system, let $C > 0$ be a time when the system is empty (we assume that the system is not always empty in $(0, C)$). Let $k(C)$ be the number of customers that have been served in $(0, C)$. In the following we set $k = k(C)$ for ease of notation. Let a_i be the arrival time of the i-th customer, and let d_i be the departure time of the i-th customer, $i = 1, 2, \ldots, k$.

These dates form an increasing sequence of times $(t_n)_{n=1}^{2k}$ such that

$$a_1 = t_1 < t_2 < \cdots < t_{2k-1} < t_{2k} = d_k.$$

The mean sojourn time $\overline{T}$ of a customer in $(0, C)$ is by definition

$$\overline{T} = \frac{1}{k} \sum_{i=1}^{k} (d_i - a_i)$$

since $d_i - a_i$ is the time spent in the system by the i-th customer.

Let us now compute $\overline{N}$, the mean number of customers in the system in $(0, C)$. Denote by $N(t)$ the number of customers at time t. Then,

$$\begin{aligned} \overline{N} &= \frac{1}{C} \int_0^C N(t)\, dt \\ &= \frac{1}{C} \sum_{i=1}^{2k-1} N(t_i^+)\,(t_{i+1} - t_i) \end{aligned}$$

where $N(t^+)$ is the number of customers in the system *just after* time t.

It is not difficult to see (make a picture) that

$$\sum_{i=1}^{k} (d_i - a_i) = \sum_{i=1}^{2k-1} N(t_i^+)\,(t_{i+1} - t_i).$$

Hence,

$$\overline{N} = \frac{k}{C}\,\overline{T}.$$

The proof is concluded as follows: since the system empties infinitely often we can choose C large enough so that k/C is equal to the arrival rate λ. Hence, $\overline{T} = \lambda\,\overline{N}$.

■

2.7 Comparing different multiprocessor systems

While designing a multiprocessor system we may wish to compare different systems.

The first system is an M/M/2 queue with arrival rate 2λ and service rate μ.

The second system is an M/M/1 queue with arrival rate 2λ and service rate 2μ.

Note that the comparison is fair since in both systems the traffic intensity denoted by ν is given by $\nu = \lambda/\mu$.

What system yields the smallest expected customer response time?

Let $\overline{T}_1$ and $\overline{T}_2$ be the expected customer response time in systems 1 and 2, respectively.

Computation of $\overline{T}_1$:

Denote $\overline{N_1}$ by the mean number of customers in the system. From (44) and (45) we get that

$$\pi(i) = 2\,\pi(0)\,\nu^i \quad \forall i \geq 1,$$

if $\nu < 1$, from which we deduce that

$$\pi(0) = \frac{1-\nu}{1+\nu}.$$

Thus, for $\nu < 1$,

$$\begin{aligned}
\overline{N}_1 &= \sum_{i=1}^{\infty} i\,\pi(i) \\
&= 2\left(\frac{1-\nu}{1+\nu}\right)\sum_{i=1}^{\infty} i\,\nu^i \\
&= \frac{2\nu}{(1-\nu)\,(1+\nu)}
\end{aligned}$$

by using the well-known identity $\sum_{i=1}^{\infty} i\, z^{i-1} = 1/(1-z)^2$ for all $0 \leq z < 1$.

From Little's formula we deduce that

$$\overline{T}_1 = \frac{\nu}{\lambda\,(1-\nu)\,(1+\nu)}$$

under the stability condition $\nu < 1$.

Computation of $\overline{T}_2$:

For the M/M/1 queue with arrival rate 2λ and service rate 2μ we have already seen in Example 2.1 (take $\rho = \nu$) that

$$\overline{T}_2 = \frac{\nu}{2\lambda\,(1-\nu)}$$

under the stability condition $\nu < 1$.

It is easily seen that $\overline{T_2} < \overline{T_1}$ when $\nu < 1$.

2.8 The M/G/1 Queue

This is a queue where the arrivals are Poisson, with rate $\lambda > 0$, and where the service times of the customers are independent with the *same*, *arbitrary*, c.d.f. $G(x)$. More precisely, if σ_i and σ_j are the service times of two customers, say customers i and j, $i \neq j$, respectively, then

(1) σ_i and σ_j are independent r.v.'s

(2) $G(x) = P(\sigma_i \leq x) = P(\sigma_j \leq x)$ for all $x \geq 0$.

Let $1/\mu$ be the mean service time, namely, $1/\mu = E[\sigma_i]$. The service times are further assumed to be independent of the arrival process.

As usual we will set $\rho = \lambda/\mu$. We will also assume that customers are served according to the service discipline First-In-First-Out (FIFO).

For this queueing system, the process $(N(t),\, t \geq 0)$, where $N(t)$ is the number of customers in the queue at time t, is *not* a Markov process. This is because the probabilistic future of $N(t)$ for $t > s$ cannot be determined if one only knows $N(s)$, except if $N(s) = 0$ (consider for instance the case when the service times are all equal to the same constant).

2.8.1 Mean Queue-Length and Mean Response Time

Define W_n to be the waiting time in queue of the n-th customer under the FIFO service discipline. Let

- $\overline{W}$ be the mean waiting time
- $X(t)$ be the number of customers in the *waiting room* at time t,
- $R(t)$ be the *residual service time* of the customer in the server at time t, if any
- t_n denote the arrival time of the n-th customer for all $n \geq 1$
- σ_n the service time of customer n. Note that $E[\sigma_n] = 1/\mu$.

We will assume by convention that $X(t_i)$ is the number of customers in the waiting room just *before* the arrival of the i-th customer. We have

$$\begin{aligned} E[W_i] &= E[R(t_i)] + E\left[\sum_{j=i-X(t_i)}^{i-1} \sigma_j\right] \\ &= E[R(t_i)] + \sum_{k=0}^{\infty} \sum_{j=i-k}^{i-1} E[\sigma_j \mid X(t_i) = k] \, P(X(t_i) = k) \\ &= E[R(t_i)] + \frac{1}{\mu} E[X(t_i)]. \end{aligned} \tag{54}$$

To derive (54) we have used the fact that σ_j is independent of $X(t_i)$ for $j = i - X(t_i), \ldots, i-1$, which implies that $E[\sigma_j \mid X(t_i) = k] = 1/\mu$. Indeed, $X(t_i)$ only depends on the service times σ_j for $j = 1, \ldots, i - X(t_i) - 1$ and not on σ_j for $j \geq i - X(t_i)$ since the service discipline is FIFO.

Letting now $i \to \infty$ in (54) yields

$$\overline{W} = \overline{R} + \frac{\overline{X}}{\mu} \tag{55}$$

with

- $\overline{R} := \lim_{i\to\infty} E[R(t_i)]$ is the mean service time at *arrival epochs* in steady-state, and
- $\overline{X} := \lim_{i\to\infty} E[X(t_i)]$ is the mean number of customers in the waiting room at *arrival epochs* in steady-state.

Because the arrival process is a Poisson process (PASTA property: Poisson Arrivals See Times Averages), we have that

$$\overline{R} = \lim_{t\to\infty} \frac{1}{t} \int_0^t R(s)\, ds \tag{56}$$

$$\overline{X} = \lim_{t\to\infty} \frac{1}{t} \int_0^t X(s)\, ds. \tag{57}$$

We shall not proved these results.[4]

In words, (56) says that the mean residual service times at *arrival* epochs and at *arbitrary* epochs are the same. Similarly, (57) expresses the fact that the mean number of customers at *arrival* epochs and at *arbitrary* epochs are the same.

Example 2.2 If the arrivals are not Poisson then formulae (56) and (57) are in general not true. Here is an example where (56) is not true: assume that the nth customer arrives at time $t_n = n$ seconds (s) for all $n \geq 1$ and that it requires 0.999s of service (i.e., $\sigma_n = 0.999$s). If the system is empty at time 0, then clearly $R(t_n) = 0$ for all $n \geq 1$ since an incoming customer will always find the system empty, and therefore the left-hand side of (56) is zero; however, since the server is always working in $(n, n + 0.999)$ for all $n \geq 1$ it should be clear that the right-hand side of (56) is $(0.999)^2/2$. ♡

Applying Little's formula to the waiting room yields

$$\overline{X} = \lambda \overline{W}$$

so that, cf. (55),

$$\overline{W}(1-\rho) = \overline{R}. \tag{58}$$

From now on we will assume that $\rho < 1$. Hence, cf. (58),

$$\overline{W} = \frac{\overline{R}}{1-\rho}. \tag{59}$$

The condition $\rho < 1$ is the stability condition of the M/G/1 queue. This condition is again very natural. We will compute $\overline{R}$ under the assumption that the queue empties infinitely often (it can be shown that this occurs with probability 1 if $\rho < 1$). Let C be a time when the queue is empty and define $Y(C)$ to be the number of customers served in $(0, C)$.

[4] The proof is not technically difficult but beyond the scope of this course; I can give you references if you are interested in.

We have (hint: display the curve $t \to R(t)$):

$$
\begin{aligned}
\overline{R} &= \lim_{C\to\infty} \frac{1}{C} \sum_{n=1}^{Y(C)} \frac{\sigma_i^2}{2} \\
&= \lim_{C\to\infty} \left(\frac{Y(C)}{C}\right) \lim_{C\to\infty} \left(\frac{1}{Y(C)} \sum_{n=1}^{Y(C)} \frac{\sigma_i^2}{2}\right) \\
&= \lambda \frac{E[\sigma^2]}{2}
\end{aligned}
$$

where $E[\sigma^2]$ is the second-order moment of the service times (i.e., $E[\sigma^2] = E[\sigma_i^2]$ for all $i \geq 1$).

Hence, for $\rho < 1$,

$$
\overline{W} = \frac{\lambda\, E[\sigma^2]}{2\,(1-\rho)}. \tag{60}
$$

This formula is the *Pollaczek-Khinchin* (abbreviated as P-K) formula for the mean waiting in an M/G/1 queue.

Thus, the mean system response time $\overline{T}$ is given by

$$
\overline{T} = \frac{1}{\mu} + \frac{\lambda\, E[\sigma^2]}{2\,(1-\rho)} \tag{61}
$$

and, by Little's formula, the mean number of customers $E[N]$ in the entire system (waiting room + server) is given by

$$
\overline{N} = \rho + \frac{\lambda^2\, E[\sigma^2]}{2\,(1-\rho)}. \tag{62}
$$

Consider the particular case when $P(\sigma_i \leq x) = 1 - \exp(-\mu x)$ for $x \geq 0$, that is, the M/M/1 queue. Since $E[\sigma^2] = 2/\mu^2$, we see from (60) that

$$
\overline{W} = \frac{\lambda}{\mu^2\,(1-\rho)} = \frac{\rho}{\mu\,(1-\rho)}
$$

which agrees with (53).

It should be emphazised that $\overline{W}$, $\overline{T}$ and $\overline{N}$ now depend upon the *first two moments* ($1/\mu$ and $E[\sigma^2]$) of the service time d.f. (and of course upon the arrival rate). This is in constrast with the M/M/1 queue where these quantities only depend upon the mean of the service time (and upon the arrival rate).

2.8.2 Mean Queue-Length at Departure Epochs

Let Q_n be the number of customers in the system *just after* the n-th departure. Let V_{n+1} be the number of customers that have arrived during the service time of the $(n+1)$-st customer, that is, during σ_{n+1}. Then, for all $n \geq 1$,

$$Q_{n+1} = \begin{cases} Q_n - 1 + V_{n+1} & \text{if } Q_n \geq 1 \\ \\ V_{n+1} & \text{if } Q_n = 0. \end{cases} \tag{63}$$

It is convenient to introduce the function Δ_k such that $\Delta_k = 1$ if $k \geq 1$ and $\Delta_0 = 0$. We may now write (63) as

$$Q_{n+1} = Q_n - \Delta_{Q_n} + V_{n+1} \quad \forall n \geq 0. \tag{64}$$

As usual, we will be concerned with $Q = \lim_{n\to\infty} Q_n$. We assume that the j-th moment of Q exists for $j = 1, 2$. Our objective is to compute $E[Q]$.

Taking the expectation in both sides of (64) yields

$$E[Q_{n+1}] = E[Q_n] - E\left[\Delta_{Q_n}\right] + E[V_{n+1}] \quad \forall n \geq 0. \tag{65}$$

Let us compute $E[V_{n+1}]$.

We have

$$\begin{aligned} E[V_{n+1}] &= \int_0^\infty E[V_{n+1} \,|\, \sigma_{n+1} = y]\, dG(y) \\ &= \int_0^\infty E[\text{no. of points in } (0, y) \text{ of a Poisson process with rate } \lambda]\, dG(y) \\ &= \int_0^\infty \lambda y \; dG(y) = \rho \end{aligned} \tag{66}$$

which does not depend on n.

Letting n go to infinity in (65) gives

$$E[Q] = E[Q] - E\left[\Delta_Q\right] + \rho.$$

which implies that

$$E\left[\Delta_Q\right] = \rho. \tag{67}$$

We have not yet computed what we are looking for, namely, $E[Q]$.

Squaring both sides of (64) and *then* taking the expectation gives

$$E[Q_{n+1}^2] = E[Q_n^2] + E[(\Delta_{Q_n})^2] + E[V_{n+1}^2] - 2\,E[Q_n\,\Delta_{Q_n}] + 2\,E[Q_n\,V_{n+1}] - 2\,E[\Delta_{Q_n}\,V_{n+1}]. \tag{68}$$

From our definition of Δ_k we observe that $(\Delta_{Q_n})^2 = \Delta_{Q_n}$ and also that $Q_n \Delta_{Q_n} = Q_n$. Therefore,

$$E[(\Delta_{Q_n})^2] = E[\Delta_{Q_n}] \tag{69}$$

and

$$E[Q_n \, \Delta_{Q_n}] = E[Q_n]. \tag{70}$$

We also observe that V_{n+1}, the number of arrival during the service of the $(n+1)$-st customer is independent of Q_n from the memoryless property of a Poisson process and from the independence assumptions.

Hence, for all $n \geq 1$,

$$\begin{aligned} E[Q_n \, V_{n+1}] &= E[Q_n] \, E[V_{n+1}] = \rho \, E[Q_n] & (71) \\ E[\Delta_{Q_n} \, V_{n+1}] &= E[\Delta_{Q_n}] \, E[V_{n+1}] = \rho \, E[\Delta_{Q_n}] & (72) \end{aligned}$$

by using (66).

Combining now (68)-(72) yields

$$E[Q_{n+1}^2] = E[Q_n^2] + E[\Delta_{Q_n}] + E[V_{n+1}^2] - 2(1-\rho) \, E[Q_n] - 2 \, \rho \, E[\Delta_{Q_n}] \tag{73}$$

Letting n go to infinity in (73) and using (67) yields, for $\rho < 1$,

$$E[Q] = \rho + \frac{\lim_{n \to \infty} E[V_{n+1}^2] - \rho}{2 \, (1-\rho)}. \tag{74}$$

It remains to compute $\lim_{n \to \infty} E[V_{n+1}^2]$.

We have:

$$E[V_{n+1}^2] = \int_0^\infty E[V_{n+1}^2 \,|\, \sigma_{n+1} = y] \, dG(y). \tag{75}$$

Let us compute $E[V_{n+1}^2 \,|\, \sigma_{n+1} = y]$. We have

$$\begin{aligned} E[V_{n+1}^2 \,|\, \sigma_{n+1} = y] &= E[(\text{no. points in } (0,y) \text{ of a Poisson process with rate } \lambda)^2] \\ &= e^{-\lambda y} \sum_{k \geq 1} k^2 \, \frac{(\lambda y)^k}{k!} \\ &= e^{-\lambda y} \sum_{k \geq 1} k \, \frac{(\lambda y)^k}{(k-1)!} = e^{-\lambda y} \sum_{k \geq 1} (k-1+1) \, \frac{(\lambda y)^k}{(k-1)!} \\ &= e^{-\lambda y} \left(\sum_{k \geq 2} \frac{(\lambda y)^k}{(k-2)!} + \sum_{k \geq 1} \frac{(\lambda y)^k}{(k-1)!} \right) \\ &= e^{-\lambda y} \left((\lambda y)^2 \, e^{\lambda y} + \lambda y \, e^{\lambda y} \right) = (\lambda y)^2 + \lambda y. \end{aligned}$$

Therefore, cf. (75),

$$\begin{aligned} E[V_{n+1}^2] &= \lambda^2 \int_0^\infty y^2\, dG(y) + \lambda \int_0^\infty y\, dG(y). \\ &= \lambda^2\, E[\sigma^2] + \rho. \end{aligned} \tag{76}$$

Finally, combining (74) and (76) gives

$$E[Q] = \rho + \frac{\lambda^2\, E[\sigma^2]}{2\,(1-\rho)} \tag{77}$$

which is the same as formula (62)! In other words, the mean number of customers at any time in an M/G/1 queue is equal to the mean number of customers at departure instants.

The stochastic process $(Q_n,\ n \geq 1)$ defined by (63) is a Markov chain. This therefore suggests that there must exist another way of computing $E[Q]$.

3 Priority Queues

In many applications it is desirable to give certain classes of customers preferential treatment; the queue is ordered and the higher priority customers are served first. In the following, we focus only on priority queues with a single server. In the following, we will only focus on priority queueing systems with a single server.

In a priority queueing system customers are divided into $K \geq 2$ *classes* numbered $1, 2, \ldots, K$. The lower the priority, the higher the class number. In other words, priority i customers are given preference over priority j customers if $i < j$. We assume that the customers within a given priority class are served in FIFO order, although this assumption need not be made in many cases (e.g., when we study queue-length processes).

There are two basic classes of priority policies: the *preemptive-resume* policies, and the *non-preemptive priority* policies. Under a preemptive-resume policy service is interrupted whenever an arriving customer has higher priority than the customer in service; the newly arrived customer begins service at once. The customer C whose service was interrupted (preempted) returns to the head of its priority class, say class j. When there are no longer customers in priority classes $1, 2, \ldots, j-1$, customer C returns to the server and resumes service at the point of interruption.

There are other variations, including *preemptive-repeat* in which a preempted customer must restart service from the beginning.

3.1 Non-Preemptive Priority

We consider a non-preemptive priority M/G/1 queue: customers of priority class k, $k = 1, 2, \ldots, K$, arrive at the system according to a Poisson process with rate λ_k and have independent service times with common d.f. $G_k(x)$, mean $1/\mu_k$, and second moment $\overline{\sigma}_k^2$. We assume that the service times and the arrival times are independent i.i.d. sequences.

Let $\rho_k := \lambda_k/\mu_k$ be the *traffic intensity* with respect to class k, $k = 1, 2, \ldots, K$. Our objective is to compute the expected values of the sojourn time of class k customers, the waiting time of class k customers, and the number of customers of class k in the system. Let

- W_k be the time spent waiting (not in service) by a customer of class k
- X_k be the number of customers of class k waiting in queue
- R be the residual service time of the customer in the server.

As before, an overline denotes expectation.

Clearly,

$$\overline{T}_k = \overline{W}_k + \frac{1}{\mu_k} \tag{78}$$

for $k = 1, 2, \ldots, K$, so we need only compute $\overline{W}_k$. By the same argument used to compute the mean waiting time in an M/G/1 queue with one class of customers (see (59)) we have for the highest priority customers

$$\overline{W}_1 = \frac{\overline{R}}{1 - \rho_1} \tag{79}$$

provided that $\rho_1 < 1$. [Hint: write $\overline{W}_1 = \overline{R} + (1/\mu_1)\, E[X_1]$ and substitute $\overline{X}_1 = \lambda_1 \overline{W}_1$ from Little's formula, from which (79) follows).

Deferring the calculation of $\overline{R}$ for the moment, let us now turn to the second highest priority. Note that $\overline{W}_2$ is the sum of four terms: the mean residual service time,$\overline{R}$, the expected time $\overline{X}_1$ (respectively, $\overline{X}_2$) to serve all customers of class 1 (respectfully, class 2) in the system when a class 2 customer arrives, and the expectation of the time Z_1 to serve all customers of class 1 that will arrive during the total waiting time of a class 2 customer. Thus,

$$\overline{W}_2 = \overline{R} + \frac{1}{\mu_1}\overline{X}_1 + \frac{1}{\mu_2}\overline{X}_2 + \frac{1}{\mu_1}\overline{Z_1} \tag{80}$$

By Little's formula

$$\begin{aligned}\overline{X}_1 &= \lambda_1 \overline{W}_1\\ \overline{X}_2 &= \lambda_2 \overline{W}_2.\end{aligned}$$

Denote by W_2 be the time spent in queue by a customer of class 2. We have

$$\begin{aligned}E[Z_1] &= \int_0^\infty E[Z_1 \mid W_2 = x]\, P(W_2 \in dx)\\ &= \int_0^\infty \lambda_1\, x\, P(W_2 \in dx)\\ &= \lambda_1 \overline{W}_2\end{aligned}$$

where (81) follows from the assumption that the arrival process of class 1 customers is Poisson with rate λ_1 (see Appendix, Result C.2).

Hence,

$$\overline{W}_2 = \frac{\overline{R} + \rho_1 \overline{W}_1}{1 - \rho_1 - \rho_2}$$

provided that $\rho_1 + \rho_2 < 1$.

Using the value of $\overline{W}_1$ obtained in (79) gives

$$\overline{W}_2 = \frac{\overline{R}}{(1-\rho_1)\,(1-\rho_1-\rho_2)} \tag{81}$$

provided that $\rho_1 + \rho_2 < 1$.

By repeating the same argument, it is easily seen that

$$\overline{W}_k = \frac{\overline{R}}{\left(1 - \sum_{j=1}^{k-1} \rho_j\right)\left(1 - \sum_{j=1}^{k} \rho_j\right)} \tag{82}$$

provided that $\sum_{j=1}^{k} \rho_j < 1$.

Hence, it is seen that the condition $\rho := \sum_{j=1}^{K} \rho_j < 1$ is the stability condition of this model. From now on we assume that $\rho < 1$.

Let us now compute $\overline{R}$, the expected residual service time. This computation is similar to the computation of the mean residual service time in a standard M/G/1 queue (see Section 2.8.1).

We will compute $\overline{R}$ under the assumption that the queue empties infinitely often (it can be shown that this occurs with probability 1 if $\rho < 1$). Let C be a time when the queue is empty and define $Y_k(C)$ to be the number of customers of class k served in $(0, C)$.

Denote by $\sigma_{k,n}$ the service time of the n-th customer of class k.

We have (hint: display the curve $t \to R(t)$):

$$\begin{aligned}
\overline{R} &= \lim_{C\to\infty} \frac{1}{C}\left(\sum_{n=1}^{Y_1(C)} \frac{\sigma_{1,n}^2}{2} + \cdots + \sum_{n=1}^{Y_K(C)} \frac{\sigma_{K,n}^2}{2}\right) \\
&= \lim_{C\to\infty}\left(\frac{Y_1(C)}{C} \times \frac{1}{Y_1(C)} \sum_{n=1}^{Y_1(C)} \frac{\sigma_{1,n}^2}{2} + \cdots + \frac{Y_K(C)}{C} \times \frac{1}{Y_K(C)} \sum_{n=1}^{Y_K(C)} \frac{\sigma_{K,n}^2}{2}\right) \\
&= \sum_{j=1}^{K}\left(\lim_{C\to\infty}\left(\frac{Y_j(C)}{C}\right) \lim_{C\to\infty}\left(\frac{1}{Y_j(C)} \sum_{n=1}^{Y_j(C)} \frac{\sigma_{j,n}^2}{2}\right)\right) \\
&= \sum_{j=1}^{K}\left(\lim_{C\to\infty}\left(\frac{Y_j(C)}{C}\right) \lim_{m\to\infty}\left(\frac{1}{m} \sum_{n=1}^{m} \frac{\sigma_{j,n}^2}{2}\right)\right) \qquad (83) \\
&= \sum_{j=1}^{K} \lambda_j \frac{\overline{\sigma_j^2}}{2}. \qquad (84)
\end{aligned}$$

where (83) follows from the fact that $Y(C) \to \infty$ when $C \to \infty$.

Finally, cf. (82), (84),

$$\overline{W}_k = \frac{\sum_{j=1}^{K} \lambda_j \overline{\sigma_j^2}}{2\left(1 - \sum_{j=1}^{k-1} \rho_j\right)\left(1 - \sum_{j=1}^{k} \rho_j\right)} \qquad (85)$$

for all $k = 1, 2, \ldots, K$, and

$$\overline{T}_k = \frac{1}{\mu_k} + \frac{\sum_{j=1}^{K} \lambda_j \overline{\sigma_j^2}}{2\left(1 - \sum_{j=1}^{k-1} \rho_j\right)\left(1 - \sum_{j=1}^{k} \rho_j\right)}. \qquad (86)$$

Applying again Little's formula to (86) gives

$$\overline{N}_k = \rho_k + \frac{\lambda_k \sum_{j=1}^{K} \lambda_j \overline{\sigma_j^2}}{2\left(1 - \sum_{j=1}^{k-1} \rho_j\right)\left(1 - \sum_{j=1}^{k} \rho_j\right)} \qquad (87)$$

for all $k = 1, 2, \ldots, K$.

3.2 Preemptive-Resume Priority

We consider the same model as in the previous section but we now assume that the policy is preemptive-resume. We keep the same notation and we assume that $\rho < 1$.

Our objective is to compute $\overline{T}_k$ and $\overline{N}_k$.

Let us first consider $\overline{T}_1$. Actually, there is nothing to compute since $\overline{T}_1$ is simply the sojourn time in an ordinary M/G/1 queue from the very definition of a preemptive-resume priority policy, that is,

$$\overline{T}_1 = \frac{1}{\mu_1} + \frac{\lambda_1 \overline{\sigma_1}^2}{2\,(1-\rho_1)}. \tag{88}$$

The sojourn of a customer a class k, $k \geq 2$, is the sum of three terms:

$$\overline{T}_k = \overline{T}_{k,1} + \overline{T}_{k,2} + \overline{T}_{k,3}$$

where

(1) $\overline{T}_{k,1}$ is the customer's average service time, namely, $\overline{T}_{k,1} = 1/\mu_k$

(2) $\overline{T}_{k,2}$ is the average time required, upon arrival of a class k customer, to service customers of class 1 to k already in the system

(3) $\overline{T}_{k,3}$ is the average sojourn time of customers of class 1 to $k-1$ who arrive while the customer of class k is in the system.

<u>Computation of $\overline{T}_{k,2}$:</u>

When a customer of class k arrives his waiting time before entering the server for the first time is the same as his waiting time in an ordinary M/G/1 queue (without priority) where customers of class $k+1, \ldots, K$ are neglected (i.e., $\lambda_i = 0$ for $i = k+1, \ldots, K$). The reason is that the sum of remaining service times of all customers in the system is *independent* of the service discipline of the system. This is true for any system where the server is always busy when the system is nonempty.

Hence,

$$\overline{T}_{k,2} = \overline{R}_k + \sum_{i=1}^{k} \frac{\overline{X}_i}{\mu_i}$$

where $\overline{R}_k$ is the residual service time, and $\overline{X}_i$ is the mean number of customers of class i in the waiting room.

By Little's formula, $\overline{X}_i = \lambda_i \, \overline{T}_{k,2}$ for $i = 1, 2, \ldots, k$.

Also, by the same argument as the one used to derive (84), we get that

$$\overline{R}_k = \sum_{i=1}^{k} \lambda_i \, \frac{\overline{\sigma_i^2}}{2}.$$

This finally yields the equation

$$\overline{T}_{k,2} = \sum_{i=1}^{k} \lambda_i \, \frac{\overline{\sigma_i^2}}{2} + \sum_{i=1}^{k} \rho_i \, \overline{T}_k^2$$

and

$$\overline{T}_{k,2} = \frac{\sum_{i=1}^{k} \lambda_i \, \frac{\overline{\sigma_i^2}}{2}}{1 - \sum_{i=1}^{k} \rho_i}.$$

Computation of $\overline{T}_{k,3}$:

By Little's formula, the average number of class i customers, $i = 1, 2, \ldots, k-1$, who arrive during the sojourn time of a class k customer is $\lambda_i \, \overline{T}_k$.

Therefore,

$$\overline{T}_{k,3} = \sum_{i=1}^{k-1} \rho_i \, \overline{T}_k.$$

Finally,

$$\overline{T}_k = \frac{1}{1 - \sum_{i=1}^{k-1} \rho_i} \left(\frac{1}{\mu_k} + \frac{\sum_{i=1}^{k} \lambda_i \, \overline{\sigma_i^2}}{2 \left(1 - \sum_{i=1}^{k} \rho_i \right)} \right) \tag{89}$$

and, by Little's formula,

$$\overline{N}_k = \frac{1}{1 - \sum_{i=1}^{k-1} \rho_i} \left(\rho_k + \frac{\lambda_k \sum_{i=1}^{k} \lambda_i \, \overline{\sigma_i^2}}{2 \left(1 - \sum_{i=1}^{k} \rho_i \right)} \right). \tag{90}$$

4 Single-Class Queueing Networks

So far, the queueing systems we studied were only *single resource* systems: that is, there was one service facility, possibly with multiple servers. Actual computer systems and communication networks are *multiple resource systems*. Thus, we may have online terminals or workstations, communication lines, etc., as well as the computer itself. The computer, even the simplest personal computer, has multiple resources, too, including main memory, virtual memory, coprocessors, I/O devices, etc. There may be a queue associated with each of these resources. Thus, a computer system or a communication network is a network of queues.

A queueing network is *open* if customers enter from outside the network, circulate among the service centers (or queues or nodes) for service, and depart from the network. A queueing network is *closed* if a *fixed* number of customers circulate indefinitely among the queues. A queueing network is *mixed* if some customers enter from outside the network and eventually leave, and if some customers always remain in the network.

4.1 Networks of Markovian Queues: Open Jackson Network

Consider an open network consisting of K M/M/1 queues. Jobs arrive from outside the system joining queue i according to a Poisson process with rate λ_i^0. After service at queue i, which is *exponentially* distributed with parameter μ_i, the job either leaves the system with probability p_{i0}, or goes to queue j, with probability p_{ij}. Clearly, $\sum_{j=0}^{K} p_{ij} = 1$, since each job must go somewhere.

As usual, the arrival times and the service times are assumed to be mutually independant r.v.'s.

Let $Q_i(t)$ be the number of customers in queue (or node) i at time t and define

$$Q(t) = (Q_1(t), \ldots, Q_K(t)) \quad \forall t \geq 0.$$

As usual we will be interested in the computation of

$$\pi(\underline{n}) = \lim_{t \to \infty} P(Q(t) = \underline{n})$$

with $\underline{n} := (n_1, \ldots, n_K) \in \mathbb{N}^K$.

Because of the Poisson and exponential assumptions the continuous-time, discrete-space stochastic process $(Q(t),\, t \geq 0)$ is seen to be a continuous-time Markov chain on the state-space $I = \mathbb{N}^K$.

The balance equations for this C-M.C. are (cf. 30)

$$\pi(\underline{n}) \left(\sum_{i=1}^{K} \lambda_i^0 + \sum_{i=1}^{K} \mathbf{1}(n_i > 0)\, \mu_i \right) \quad = \quad \sum_{i=1}^{K} \mathbf{1}(n_i > 0)\, \lambda_i^0\, \pi(\underline{n} - \underline{e}_i)$$

$$+\sum_{i=1}^{K} p_{i0}\,\mu_i\,\pi(\underline{n}+\underline{e}_i)$$

$$+\sum_{i=1}^{K}\sum_{j=1}^{K} \mathbf{1}(n_j>0) p_{ij}\mu_i\,\pi(\underline{n}+\underline{e}_i-\underline{e}_j) \qquad (91)$$

where $\mathbf{1}(k>0)=1$ if $k>0$ and 0 otherwise, and where $\underline{e}_i$ is the vector with all components zero, except the i-th one which is one.

Result 4.1 (Open Jackson network) *If $\lambda_i < \mu_i$ for all $i=1,2,\ldots,K$, then*

$$\pi(\underline{n}) = \prod_{i=1}^{K}\left(1-\frac{\lambda_i}{\mu_i}\right)\left(\frac{\lambda_i}{\mu_i}\right)^{n_i} \qquad \forall \underline{n}=(n_1,\ldots,n_K)\in \mathbb{N}^K, \qquad (92)$$

where $(\lambda_1,\lambda_2\ldots,\lambda_K)$ is the unique nonnegative solution of the system of linear equations

$$\lambda_i = \lambda_i^0 + \sum_{j=1}^{K} p_{ji}\,\lambda_j \quad i=1,2,\ldots,K. \qquad (93)$$

□

Let us comment this fundamental result of queueing network theory obtained by J. R. Jackson in 1957.

Equations (93) are referred to as the traffic equations. Let us show that λ_i is the total arrival rate at node i when the system is in steady-state.

To do so, let us first determine the total throughput of a node. The total throughput of node i consists of the customers who arrive from outside the network with rate λ_i^0, plus all the customers who are transferred to node i after completing service at node j for all nodes in the network. If λ_i is the total throughput of node i, then the rate at which customers arrive at node i from node j is $p_{ji}\lambda_j$. Hence, the throughput of node i, λ_i must satisfy (93).

Since, in steady-state, the throughput of every node is equal to the arrival rate at this node, we see that λ_i is also the total arrival rate in node i.

Hence, the conditions $\lambda_i < \mu_i$ for $i=1,2,\ldots,K$, are the stability conditions of an open Jackson network.

Let us now discuss the form of the limiting distribution (92). We see that (92) is actually a product of terms, where the i-th term $(1-\rho_i)\,\rho_i^{n_i}$, with $\rho_i := \lambda_i/\mu_i$, is actually the limiting d.f. of the number of customers in an isolated M/M/1 queue with arrival rate λ_i and service rate μ_i. This property is usually referred to as the product-form property.

Therefore, the network state probability (i.e., $\pi(\underline{n})$) is the product of the state probabilities of the individual queues.

It is important to note that the steady-state probabilities behave as if the total arrival process at every node (usually referred to as the *flow*) were Poisson (with rate λ_i for node i), but the flows are not Poisson in general !! The flows are Poisson if and only if $p_{i,i+1} = 1$ for $i = 1, 2, \ldots, K$, and $p_{K0} = 1$, that is, for a network of queues in <u>series</u>.

Proof of Result 4.1: Since the network is open we know that there exists i_0 such that $p_{i_0 0} > 0$ since otherwise customers would stay forever in the network. This ensures that the matrix $I - P$ is invertible, with $P = [p_{ij}]_{1 \leq i,j \leq K}$, from which we may deduce that the balance equations (91) have a unique nonnegative solution $(\lambda_1, \ldots, \lambda_K)$ (this proof is omitted).

On the other hand, because every node may receive and serve infinitely many customers, and because of the markovian assumptions it is seen that the C-M.C. $(Q(t), t \geq 0)$ is irreducible (hint: one must show that the probability of going from state $(n_1, \ldots, n_K)$ to state $(m_1, \ldots, m_K)$ in exactly t units of time is strictly positive for all $t > 0$, $n_i \geq 0$, $m_i \geq 0$, $i = 1, 2, \ldots, K$. Let $s < t$. Since the probability of having 0 external arrival in $[0, s)$ is stricltly positive we see that the probability that the system is empty at time s is also strictly positive. On the other hand, starting from an empty system, it should be clear that we can reach any state in exactly $t - s$ units of time).

Hence, Result 1.5 applies to the irreducible C.-M.C. $(Q(t), t \geq 0)$. Thus, it suffices to check that (92) satisfies the balance equations together with the condition $\sum_{\underline{n} \in \mathbb{N}^K} \pi(\underline{n}) = 1$.

Observe from (92) that the latter condition is trivially satisfied. It is not difficult to check by direct substitution that (92) indeed satisfies the balance equations (30). This concludes the proof. ∎

This result actually extends to the case when the network consists of K M/M/c queues. Assume that node i is an M/M/c_i queue. The following result holds:

Result 4.2 (Open Jackson network of M/M/c queues) *Define* $\mu_i(r) = \mu_i \min(r, c_i)$ *for* $r \geq 0$, $i = 1, 2, \ldots, K$ *and let* $\rho_i = \lambda_i / \mu_i$ *for* $i = 1, 2, \ldots, K$.

If $\lambda_i < c_i \, \mu_i$ *for all* $i = 1, 2, \ldots, K$, *then*

$$\pi(\underline{n}) = \prod_{i=1}^{K} C_i \left(\frac{\lambda_i^{n_i}}{\prod_{r=1}^{n_i} \mu_i(r)} \right) \quad \forall \underline{n} = (n_1, \ldots, n_K) \in \mathbb{N}^K, \tag{94}$$

where $(\lambda_1, \lambda_2 \ldots, \lambda_K)$ *is the unique nonnegative solution of the system of linear equations*

(93), and where C_i is given by

$$C_i = \left[\sum_{r=0}^{c_i - 1} \frac{\rho_i^r}{r!} + \left(\frac{\rho_i^{c_i}}{c_i!} \right) \left(\frac{1}{1 - \rho_i / c_i} \right) \right]^{-1}. \tag{95}$$

$\square$

Example 4.1 Consider a switching facility that transmits messages to a required destination. A NACK (Negative ACKnowledgment) is sent by the destination when a packet has not been properly transmitted. If so, the packet in error is retransmitted as soon as the NACK is received.

We assume that the time to send a message and the time to receive a NACK are both exponentially distributed with parameter μ. We also assume that packets arrive at the switch according to a Poisson process with rate λ^0. Let p, $0 < p \leq 1$, be the probability that a message is received correctly.

Thus, we can model this switching facility as a Jackson network with one node, where $c_1 = 1$ (one server), $p_{10} = p$ and $p_{11} = 1 - p$. By Jackson's theorem we have that, $\pi(n)$, the number of packets in the service facility in steady-state, is given by

$$\pi(n) = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n \quad n \geq 0,$$

provided that $\lambda < \mu$, where λ is the solution of the traffic equation

$$\lambda = \lambda^0 + (1 - p)\lambda.$$

Therefore, $\lambda = \lambda^0 / p$, and

$$\pi(n) = \left(1 - \frac{\lambda^0}{p\mu}\right) \left(\frac{\lambda^0}{p\mu}\right)^n \quad n \geq 0,$$

provided that $\lambda^0 < p\mu$.

The mean number of packets (denoted as $\overline{X}$) in the switching facility is then given by (see (40))

$$\overline{X} = \frac{\lambda^0}{p\mu - \lambda^0}$$

and by, Little's formula, the mean response time (denoted as $\overline{T}$) is

$$\overline{T} = \frac{1}{p\mu - \lambda^0}.$$

♡

Example 4.2 We consider the model in Example 4.1 but we now assume that the switching facility is composed of K nodes in series, each modeled as $M/M/1$ queue with common service rate μ. In other words, we now have a Jackson network with K M/M/1 queues where $\lambda_i^0 = 0$ for $i = 2, 3, \ldots, K$ (no external arrivals at nodes $2, \ldots, K$), $\mu_i = \mu$ for $i = 1, 2, \ldots, K$, $p_{ii+1} = 1$ for $i = 1, 2, \ldots, K-1$, $p_{K,0} = p$ and $p_{K,1} = 1 - p$.

For this model, the traffic equations read

$$\lambda_i = \lambda_{i-1}$$

for $i = 2, 3, \ldots, K$, and

$$\lambda_1 = \lambda^0 + (1-p)\,\lambda_K.$$

It is easy to see that the solution to this system of equations is

$$\lambda_i = \frac{\lambda^0}{p} \quad \forall i = 1, 2, \ldots, K.$$

Hence, by Jackson's theorem, the j.d.f. $\pi(\underline{n})$ of the number of packets in the system is given by

$$\pi(\underline{n}) = \left(\frac{p\mu - \lambda^0}{p\mu}\right)^K \left(\frac{\lambda^0}{p\mu}\right)^{n_1+\cdots+n_K} \qquad \forall \underline{n} = (n_1, n_2, \ldots, n_K) \in \mathbb{N}^K$$

provided that $\lambda^0 < p\mu$. In particular, the probability $q_{ij}(r,s)$ of having r packets in node i and s packets in node $j > i$ is given by

$$\begin{aligned} q_{ij}(r,s) &= \sum_{n_l \geq 0,\ l \notin \{i,j\}} \pi(n_1, \ldots, n_{i-1}, r, n_{i+1}, \ldots, n_{j-1}, s, n_{j+1}, \ldots, n_K) \\ &= \left(\frac{p\mu - \lambda^0}{p\mu}\right)^2 \left(\frac{\lambda^0}{p\mu}\right)^{r+s}. \end{aligned}$$

Let us now determine for this model the expected sojourn time of a packet. Since queue i has the same characteristics as an M/M/1 queue with arrival rate λ^0/p and mean service time $1/\mu$, the mean number of packets (denoted as $\overline{X_i}$) is given by

$$\overline{X_i} = \frac{\lambda^0}{p\mu - \lambda^0}$$

for every $i = 1, 2, \ldots, K$. Therefore, the total expected number of packets in the network is

$$\sum_{i=1}^{K} E[X_i] = \frac{K\,\lambda^0}{p\mu - \lambda^0}.$$

Hence, by Little's formula, the expected sojourn time is given by

$$\overline{T} = \frac{1}{\lambda_0} \sum_{i=1}^{K} E[X_i] = \frac{K}{p\mu - \lambda^0}.$$

♡

Example 4.3 (The open central server network) Consider a computer system with one CPU and several I/O devices. A job enters the system from the outside and then waits until its execution begins. During its execution by the CPU, I/O requests may be needed. When an I/O request has been fulfilled the job then returns to the CPU for additional treatment. If the latter is available then the service begins at once; otherwise the job must wait. Eventually, the job is completed (no more I/O requests are requested) and it leaves the system.

We are going to model this system as an open Jackson network with 3 nodes: one node (node 1) for the CPU and two nodes (nodes 2 and 3) for the I/O devices. In other words, we assume that $K = 3$, $\lambda_i^0 = 0$ for $i = 2, 3$ (jobs cannot access the I/O devices directly from the outside) and $p_{21} = p_{31} = 1$, $p_{10} > 0$.

For this system the traffic equations are:

$$\begin{aligned}
\lambda_1 &= \lambda_1^0 + \lambda_2 + \lambda_3 \\
\lambda_2 &= \lambda_1\, p_{12} \\
\lambda_3 &= \lambda_1\, p_{13}.
\end{aligned}$$

The solution of the traffic equations is $\lambda_1 = \lambda_1^0/p_{10}$, $\lambda_i = \lambda_1^0\, p_{1i}/p_{10}$ for $i = 2, 3$. Thus,

$$\pi(\underline{n}) = \left(1 - \frac{\lambda_1^0}{\mu_1\, p_{10}}\right)\left(\frac{\lambda_1^0}{\mu_1\, p_{10}}\right)^{n_1} \prod_{i=2}^{3}\left(1 - \frac{\lambda_1^0\, p_{1i}}{\mu_i\, p_{10}}\right)\left(\frac{\lambda_1^0\, p_{1i}}{\mu_i\, p_{10}}\right)^{n_i} \qquad \forall \underline{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$$

and

$$\overline{T} = \frac{1}{\mu_1\, p_{10} - \lambda_1^0} + \sum_{i=2}^{3} \frac{p_{1i}}{\mu_i\, p_{10} - \lambda_1^0\, p_{1i}}.$$

♡

4.2 Networks of Markovian Queues: Closed Jackson Networks

We now discuss *closed* markovian queueing networks. In such networks the number of customers in the network is always constant: no customer may enter from the outside and no customer may leave the network. More precisely, a closed Jackson is an open Jackson network where $\lambda_i^0 = 0$ for $i = 1, 2, \ldots, K$. However, because the number of customers is constant in a closed Jackson network– and assumed equal to N – a particular treatment is needed. Indeed, letting $\lambda_i^0 = 0$ for $i = 1, 2, \ldots, K$ in Result 4.1 does not yield the correct result (see 4.3 below).

Without loss of generality we shall assume that each node in the network is visited infinitely often by the customers (simply remove the nodes that are only visited a finite number of times).

For this model, the balance equations are

$$\pi(\underline{n}) \sum_{i=1}^{K} \mathbf{1}(n_i > 0)\, \mu_i = \sum_{i=1}^{K} \sum_{j=1}^{K} \mathbf{1}(n_i \leq N-1,\, n_j > 0) p_{ij} \mu_i \, \pi(\underline{n} + \underline{e}_i - \underline{e}_j) \tag{96}$$

for all $\underline{n} = (n_1, \ldots, n_K) \in \{0, 1, \ldots, N\}^K$ such that $\sum_{i=1}^{K} n_i = N$.

Result 4.3 (Closed Jackson network) *Let $(\lambda_1, \ldots, \lambda_K)$ be any non-zero solution of the equations*

$$\lambda_i = \sum_{j=1}^{K} p_{ji}\, \lambda_j \quad i = 1, 2, \ldots, K. \tag{97}$$

Then,

$$\pi(\underline{n}) = \frac{1}{G(N,K)} \prod_{i=1}^{K} \left(\frac{\lambda_i}{\mu_i} \right)^{n_i} \tag{98}$$

for all $\underline{n} = (n_1, \ldots, n_K) \in S(N,K)$, with

$$S(N,K) := \left\{ \underline{n} = (n_1, \ldots, n_K) \in \{0, 1, \ldots, N\}^K \text{ such that } \sum_{i=1}^{K} n_i = N \right\}$$

where

$$G(N,K) := \sum_{\underline{n} \in S(N,K)} \prod_{i=1}^{K} \left(\frac{\lambda_i}{\mu_i} \right)^{n_i}. \tag{99}$$

□

The constant $G(N,K)$ is called the *normalization constant*. It has been named like this since it ensures that

$$\sum_{\underline{n}\in S(N,K)} \pi(\underline{n}) = 1.$$

Unlike the corresponding result for the open Jackson result, (98) shows that the number of customers in two different nodes in steady-state are <u>not</u> independent r.v.'s. This follows from the fact that the right-hand side of (98) does not write as a product of terms of the form $f_1(n_1) \times \cdots \times f_K(n_K)$.

This result is obvious: assume that there are n_i customers in node i. Then, the number of customers in node j is necessarily less than or equal to $M - n_i$. Thus, the r.v.'s X_i and X_j representing the number of customers in nodes i and j, respectively, cannot be independent (take $K = 2$; then $P(X_j = M) = 0$ if $X_i = 1$ whereas $P(X_j = M) = 1$ if $X_i = 0$).

Nevertheless, and with a sligh abuse of notation, (98) is usually referred to as a product-form theorem for closed Jackson networks.

Proof of Result 4.3: Check that (98) satisfies (96) and apply Result 1.5. ∎

Like for open Jackson networks, Result 4.3 extends to closed Jackson network of M/M/c queues (we keep the notation introduced earlier) We have:

Result 4.4 (Closed Jackson network of M/M/c queues) *Let $(\lambda_1, \ldots, \lambda_K)$ be any non-zero solution to the traffic equations (97). Then,*

$$\pi(\underline{n}) = \frac{1}{G(N,K)} \prod_{i=1}^{K} \left(\frac{\lambda_i^{n_i}}{\prod_{r=1}^{n_i} \mu_i(r)} \right) \tag{100}$$

for all $\underline{n} = (n_1, \ldots, n_K) \in S(N,K)$, where

$$G(N,K) := \sum_{\underline{n}\in S(N,K)} \prod_{i=1}^{K} \left(\frac{\lambda_i^{n_i}}{\prod_{r=1}^{n_i} \mu_i(r)} \right). \tag{101}$$

□

The computation of the normalizing constant $G(N,K)$ seems to be an easy task: it suffices to add a bunch of terms and to do a couple of multiplications. Yes? Well...

Assume that $K = 5$ (five nodes) and $N = 10$ (10 customers). Then, the set $S(5,10)$ already contains 1001 elements. If If $K = 10$ (ten queues) and $N = 35$ customers, then $S(10,35)$

contains 52,451,256 terms!! In addition, each term requires the computation of ten constants so that the total number of multiplications is over half a billion! (A communication network may have several hundredths of nodes...) More generally, it is not difficult to see that $S(N,K)$ has $\binom{N+K-1}{N}$ elements.

Therefore, a brute force approach (direct summation) may be both very expensive and numerically unstable.

There exist stable and efficient algorithms to compute $G(N,K)$. The first algorithm was obtained by J. Buzen in 1973 and almost every year one (or sometimes several) new algorithms appear! (for, however, more general "product-form"queueing networks to be described later on in this course).

4.2.1 The Convolution Algorithm

In this section we present the so-called *convolution algorithm* for the computation of $G(N,K)$. It is due to J. Buzen.

By definition

$$\begin{aligned}
G(n,m) &= \sum_{\underline{n}\in S(n,m)} \prod_{i=1}^{m} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i} \\
&= \sum_{k=0}^{n} \sum_{\substack{\underline{n}\in S(n,m)\\ n_m=k}} \prod_{i=1}^{m} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i} \\
&= \sum_{k=0}^{n} \left(\frac{\lambda_m}{\mu_m}\right)^{k} \sum_{\underline{n}\in S(n-k,m-1)} \prod_{i=1}^{m-1} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i} \\
&= \sum_{k=0}^{n} \left(\frac{\lambda_m}{\mu_m}\right)^{k} G(n-k,m-1). \qquad (102)
\end{aligned}$$

From the definition of $G(n,m)$ the initial conditions for the algorithm are

$$\begin{aligned}
G(n,1) &= \left(\frac{\lambda_1}{\mu_1}\right)^{n} \quad \text{for } n=0,1,\ldots,N \\
G(0,m) &= 1 \quad \text{for } m=1,2,\ldots,K.
\end{aligned}$$

This convolution-like expression accounts for the name "convolution algorithm".

A similar algorithm exists for a closed Jackson network with M/M/c queues.

4.2.2 Performance Measures from Normalization Constants

Performance evaluation can be expressed as functions of the equilibrium state probabilities. Unfortunately, this approach can lead to the same problems of excessive and inaccurate computations that were encountered in the calculation of the normalization constant. Fortunately, a number of important performance measures can be computed as functions of the various normalization constants which are a product of the convolution algorithm. In this section, it will be shown how this can be done.

<u>Marginal Distribution of Queue-Length</u>

Denote by X_i the number of customers in node i in steady-state. Define $\pi_i(k) = P(X_i = k)$ be the steady-state p.f. of X_i.

We have

$$\pi_i(k) = \sum_{\underline{n} \in S(N,K),\, n_i = k} \pi(\underline{n}).$$

To arrive at the marginal distribution π_i it will be easier to first calculate

$$\begin{aligned}
P(X_i \geq k) &= \sum_{\substack{\underline{n} \in S(N,K) \\ n_i \geq k}} \pi(\underline{n}) \\
&= \sum_{\substack{\underline{n} \in S(N,K) \\ n_i \geq k}} \frac{1}{G(N,K)} \prod_{j=1}^{K} \left(\frac{\lambda_j}{\mu_j}\right)^{n_j}. \\
&= \left(\frac{\lambda_i}{\mu_i}\right)^k \frac{1}{G(N,K)} \sum_{\underline{n} \in S(N-k,K)} \prod_{j=1}^{K} \left(\frac{\lambda_j}{\mu_j}\right)^{n_j} \\
&= \left(\frac{\lambda_i}{\mu_i}\right)^k \frac{G(N-k,K)}{G(N,K)}.
\end{aligned} \tag{103}$$

Now the key to calculating the marginal distribution is to recognize that

$$P(X_i = k) = P(X_i \geq k) - P(X_i \geq k+1)$$

so that

$$\pi_i(k) = \left(\frac{\lambda_i}{\mu_i}\right)^k \frac{1}{G(N,K)} \left[G(N-k,K) - \left(\frac{\lambda_i}{\mu_i}\right) G(N-k-1,K)\right]. \tag{104}$$

<u>Expected Queue-Length</u>

Perhaps the most useful statistic that can be derived from the marginal queue-length distribution is its mean. Recall the well-known formula for the mean of a r.v. X with values in

$\mathbb{N}$:

$$E[X] = \sum_{k \geq 1} P(X \geq k).$$

Therefore, cf. (103),

$$E[X_i] = \sum_{k=1}^{N} \left(\frac{\lambda_i}{\mu_i}\right)^k \frac{G(N-k,K)}{G(N,K)}. \tag{105}$$

<u>Utilization</u>

The utilization of node i — denoted as U_i — is defined to be the probability that node i is non-empty in steady-state, namely, $U_i = 1 - \pi_i(0)$.

From (104), we have that

$$U_i = \left(\frac{\lambda_i}{\mu_i}\right) \frac{G(N-1,K)}{G(N,K)}. \tag{106}$$

<u>Throughput</u>

The throughput T_i of node i is defined as

$$T_i = \sum_{k=1}^{N} \pi_i(k)\, \mu_i$$

or, equivalently, $T_i = \mu_i\,(1 - \pi_i(0)) = \mu_i\, U_i$.

Therefore, cf. (106),

$$T_i = \lambda_i \frac{G(N-1,K)}{G(N,K)}. \tag{107}$$

4.3 The Central Server Model

This is one a very useful model. There is one CPU (node M+1), K I/O devices (nodes M+2,...,M+K+1), and M terminals (nodes 1,2,...,K) that send jobs to the CPU.

When a user (i.e., terminal) has sent a job to the CPU he waits the end of the execution of his job before sending a new job. (we could also consider the case when a user is working on several jobs at a time). Hence, there are exactly K jobs in the network. This is a closed network.

For the time being, we are going to study a variant of this system. The central server system will be studied later on in this course.

We assume that

$$\begin{aligned}
p_{iM+1} &= 1 \quad \text{for } i = M+2, \ldots, M+K+1 \\
p_{iM+1} &= 1 \quad \text{for } i = 1, 2, \ldots, M \\
p_{M+1i} &> 0 \quad \text{for } i = 1, 2, \ldots, M+K+1 \\
p_{ij} &= 0 \quad \text{otherwise} .
\end{aligned} \tag{108}$$

Note here that several jobs may be waiting at a terminal.

Let μ_i be the service rate at node i for $i = 1, 2, \ldots, M+K+1$. For $i = 1, 2, \ldots, M$, $1/\mu_i$ can be thought of as the mean *thinking* time of the user before sending a new job to the CPU.

For this model, the traffic equations are:

$$\lambda_i = \begin{cases} \lambda_{M+1}\, p_{M+1i} & \text{for } i \neq M+1 \\ \\ \sum_{j=1}^{M+K+1} \lambda_j\, p_{jM+1} & \text{for } i = M+1 \end{cases}$$

Setting (for instance) $\lambda_{M+1} = 1$ yields $\lambda_i = p_{M+1i}$ for all $i = 1, 2, \ldots, M+K+1$, $i \neq M+1$.

The mean performance measures $E[X_i]$, U_i and T_i for $i = 1, 2, \ldots, M+K+1$, then follow from the previous section.

5 Multiclass Queueing Networks

We are now going to consider more general queueing networks that still enjoy the product-form property.

5.1 Multiclass Open/Closed/Mixed Jackson Queueing Networks

The class of systems under consideration contains an arbitrary but finite number of nodes K. There is an arbitrary but finite number R of different classes of customers. Customers travel through the network and change class according to transition probabilities. Thus a customer of class r who completes service at node i will next require service at node j in class s with a certain probability denoted $p_{i,r;j,s}$.

We shall say that the pairs (i, r) and (j, s) belong to the same subchain if the same customer can visit node i in class r and node j in class s. Let m be the number of subchains, and let $E_1, \ldots, E_m$ be the sets of states in each of these subchains.

Let n_{ir} be the number of customers of class r at node i. A *closed* system is characterized by

$$\sum_{(i,r)\in E_j} n_{ir} = \text{constant}$$

for all $j = 1, 2, \ldots, m$. In other words, if the system is closed, then there is a constant number of customer circulating in all the subchains.

In an *open* system, customers may arrive to the network from the outside according to independent Poisson process. Let λ_{ir}^0 be the external arrival rate of customers of class r at node i. In an open network a customer of class r who completes service at node i may leave the system. This occurs with the probability $p_{i,r;0}$, so that $\sum_{j,s} p_{i,r;j,s} + p_{i,r;0} = 1$.

A subchain is said to be open if it contains at least one pair (i, r) such that $\lambda_{ir}^0 > 0$; otherwise the subchain is closed. A network that contains at least one open subchain and one closed subchain is called a *mixed* network.

At node i, $i = 1, 2, \ldots, K$, the service times are still assumed to be independent exponential r.v.'s, all with the same parameter μ_i, for all $i = 1, 2, \ldots, K$. We shall further assume that the service discipline at each node is FIFO (more general nodes will be considered later on).

Define $Q(t) = (Q_1(t), \ldots, Q_K(t))$ with $Q_i(t) = (Q_{i1}(t), \ldots, Q_{iR}(t))$, where $Q_{ir}(t)$ denotes the number of customers of class r in node i at time t.

The process $(Q(t), t \geq 0)$ is not a continuous-time Markov chain because the class of a customer leaving a node is not known.

Define $X_i(t) = (I_{i1}(t), \ldots, I_{iQ_i(t)}(t))$, where $I_{ij}(t) \in \{1, 2, \ldots, R\}$ is the class of the customer in position j in node i at time t. Then, the process $((X_1(t), \ldots, X_K(t)),\, t \geq 0)$ is a C.-M.C.

We can write the balance equations (or, equivalently, the infinitesimal generator) corresponding to this C.-M.C. (this is tedious but not difficult) and obtain a product-form solution. By aggregating states we may obtain from this result the limiting j.d.f. of $(Q_1(t), \ldots, Q_K(t))$, denoted as usual by $\pi(\cdot)$.

The result is the following:

Result 5.1 (Multiclass Open/Closed/Mixed Jackson Network) *For $k \in \{1, \ldots, m\}$ such that E_k is an open subchain, let $(\lambda_{ir})_{(i,r)\in E_k}$ be the unique strictly positive solution of the traffic equations*

$$\lambda_{ir} = \lambda_{ir}^0 + \sum_{(j,s)\in E_k} \lambda_{js}\, p_{j,s;i,r} \quad \forall (i, r) \in E_k.$$

For every k in $\{1,2,\ldots,m\}$ such that E_k is a closed subchain, let $(\lambda_{ir})_{(i,r)\in E_k}$ be any non-zero solution of the equations

$$\lambda_{ir} = \sum_{(j,s)\in E_k} \lambda_{js}\, p_{j,s;i,r} \quad \forall (i,r)\in E_k.$$

If $\sum_{\{r\,:\,(i,r) \text{ belongs to an open subchain}\}} \lambda_{ir} < \mu_i$ for all $i = 1,2,\ldots,K$ (stability condition[5]), then

$$\pi(\underline{n}) = \frac{1}{G}\prod_{i=1}^{K}\left[n_i!\prod_{r=1}^{R}\frac{1}{n_{ir}!}\left(\frac{\lambda_{ir}}{\mu_i}\right)^{n_{ir}}\right] \tag{109}$$

for all $\underline{n} = (\underline{n}_1,\ldots,\underline{n}_K)$ in the state-space, where $\underline{n}_i = (n_{i1},\ldots,n_{iR}) \in \mathbb{N}^R$ and $n_i = \sum_{r=1}^{R} n_{ir}$. Here, G is a normalizing constant.

□

Let us illustrate this result through a simple example.

Example 5.1 There are two nodes (node 1 and node 2) and two classes of customers (class 1 and class 2). There are no external arrivals at node 2. External customers enter node 1 in class 1 with the rate λ. Upon service completion at node 1 a customer of class 1 is routed to node 2 with the probability 1. Upon service completion at node 2 a customer of class 1 leaves the system with probability 1.

There are always exactly K customers of class 2 in the system. Upon service completion at node 1 (resp. node 2) of customer of class 2 is routed back to node 2 (resp. node 1) in class 2 with the probability 1.

Let μ_i be the service rate at node $i = 1, 2$.

The state-space S for this system is

$$S = \left\{(n_{11}, n_{12}, n_{21}, n_{22}) \in \mathbb{N}^4 \,:\, n_{11} \geq 0, n_{21} \geq 0, n_{12} + n_{22} = K\right\}.$$

There are two subchains, E_1 and E_2, one open (say E_1) and one closed. Clearly, $E_1 = \{(1,1),(2,1)\}$ and $E_2 = \{(1,2),(2,2)\}$.

We find: $\lambda_{11} = \lambda_{21} = \lambda$ and $\lambda_{12} = \lambda_{22}$. Take $\lambda_{12} = \lambda_{22} = 1$, for instance.

[5] Observe that this condition is automatically satisfied for node i if this node is only visited by customers belonging to closed subchains.

The product-form result is:

$$\pi(\underline{n}) = \frac{1}{G} \binom{n_1}{n_{11}} \binom{n_2}{n_{22}} \left(\frac{\lambda}{\mu_1}\right)^{n_{11}} \left(\frac{\lambda}{\mu_2}\right)^{n_{21}} \left(\frac{1}{\mu_1}\right)^{n_{12}} \left(\frac{1}{\mu_2}\right)^{n_{22}}$$

with $\underline{n} \in S$, provided that $\lambda < \mu_i$ for $i = 1, 2$, i.e., $\lambda < \min(\mu_1, \mu_2)$ (stability condition).

Let us compute the normalizing constant G. By definition,

$$\begin{aligned}
G &= \sum_{\substack{n_{11}\geq 0, n_{21}\geq 0 \\ n_{12}+n_{22}=K}} \left(\frac{\lambda}{\mu_1}\right)^{n_{11}} \left(\frac{\lambda}{\mu_2}\right)^{n_{21}} \left(\frac{1}{\mu_1}\right)^{n_{12}} \left(\frac{1}{\mu_2}\right)^{n_{22}} \\
&= \left(\sum_{n_{11}\geq 0} \left(\frac{\lambda}{\mu_1}\right)^{n_{11}}\right) \left(\sum_{n_{21}\geq 0} \left(\frac{\lambda}{\mu_2}\right)^{n_{21}}\right) \sum_{n_{12}+n_{22}=K} \left(\frac{1}{\mu_1}\right)^{n_{12}} \left(\frac{1}{\mu_2}\right)^{n_{22}} \\
&= \left(\prod_{i=1}^{2} \frac{\mu_i}{\mu_i - \lambda}\right) \left(\frac{1}{\mu_1}\right)^K \sum_{i=0}^{K} \left(\frac{\mu_1}{\mu_2}\right)^i .
\end{aligned}$$

Thus,

$$G = \frac{K+1}{\mu^K} \left(\frac{\mu}{\mu - \lambda}\right)^2$$

if $\mu_1 = \mu_2 := \mu$, and

$$G = \left(\prod_{i=1}^{2} \frac{\mu_i}{\mu_i - \lambda}\right) \left(\frac{1}{\mu_1}\right)^K \frac{1 - (\mu_1/\mu_2)^{K+1}}{1 - (\mu_1/\mu_2)}$$

if $\mu_1 \neq \mu_2$.

If $K = 0$, then $G = \prod_{i=1}^{2} (\mu_i/(\mu_i - \lambda))$ as expected (open Jackson network with two M/M/1 nodes in series).

The product-form result in Result 5.1 can be dramatically extended. We now give two simple extensions.

The first, not really surprising extension owing to what we have seen before, is the extension of the product-form result to multiclass open/closed/mixed Jackson networks with M/M/c queues. Let $c_i \geq 1$ be the number of servers at node i, and define $\alpha_i(j) = \min(c_i, j)$ for all $i = 1, 2, \ldots, K$. Hence, $\mu_i \, \alpha_i(j)$ is the service rate in node i when there are j customers[6].

[6]Actually, (110) and (111) hold for any mapping $\alpha_i : \mathbb{N} \to [0, \infty)$ such that $\alpha_i(0) = 0$.

Then, the product-form result (109) becomes (the stability condition is unchanged except that μ_i must be replaced by $\mu_i c_i$):

$$\pi(\underline{n}) = \frac{1}{G} \prod_{i=1}^{K} \left[\left(\prod_{j=1}^{n_i} \frac{1}{\alpha_i(j)} \right) n_i! \left(\prod_{r=1}^{R} \frac{1}{n_{ir}!} \left(\frac{\lambda_{ir}}{\mu_i} \right)^{n_{ir}} \right) \right]. \tag{110}$$

The second extension is maybe more interesting. We may allow for *state depending* external arrival rates. Let us first introduce some notation: when the network is in state $\underline{n}$ let $M(\underline{n})$ be the total number of customers in the network, that is, $M(\underline{n}) = \sum_{i=1}^{K} n_i$.

We shall assume that the external arrival rate of customers of class r at node i maybe a function of the total number of customers in the network. More precisely, we shall assume that the external arrival rate of customers of class r at node i when the system is in state $\underline{n}$ is of the form $\lambda_{ir}^{0}\gamma(M(\underline{n}))$, where γ is an arbitrary function from $\mathbb{N}$ into $[0, \infty)$.

We have the following result:

The product-form property is preserved in a multiclass open/closed/mixed Jackson network with M/M/c nodes and state-dependent external arrivals as defined above, and is given by

$$\pi(\underline{n}) = \frac{d(\underline{n})}{G} \prod_{i=1}^{K} \left[\left(\prod_{j=1}^{n_i} \frac{1}{\alpha_i(j)} \right) n_i! \left(\prod_{r=1}^{R} \frac{1}{n_{ir}!} \left(\frac{\lambda_{ir}}{\mu_i} \right)^{n_{ir}} \right) \right]. \tag{111}$$

for all $\underline{n}$ in the state-space, where

$$d(\underline{n}) = \prod_{j=0}^{M(\underline{n})-1} \gamma(j),$$

where $d(\underline{n}) = 1$ if the network is closed.

This extension is particularly appealing in the context of communication networks since it may be used to model flow control mechanisms, namely, mechanisms that prevent congestion by denying access of packets to the network when it is too loaded.

This result may be further extended to the case when the external arrival rate of customers of class r at node i depends on the current total number of customers in the subchain that contains (i, r).

So far, only networks composed of M/M/c queues have been considered. Such nodes have obvious limitations in terms of the systems they can model:

- the service times must be exponential;
- the service times must all have the same d.f., namely, there are all exponential with the *same* parameter;
- the service discipline must be First-Come-First-Served (FCFS).

We now introduce three new types of nodes that will preserve the product-form result: Processor Sharing (PS), Last-Come-First-Served (LCFS) and the Infinite Servers (IS) nodes.

5.2 The Multiclass PS Queue

There are R classes of customers. Customers of class r enter the network according to a Poisson process with rate λ_r and require an exponential amount of service time with parameter μ_r. The K Poisson processes and service time processes are all assumed to be mutually independent processes.

There is a single server equipped with an infinite buffer. At any time. the server delivers service to customer of class r at a rate of $\mu_r\, n_r/|\underline{n}|$ when there are n_1 customers of class 1, ..., n_R customers of class R in the system, for $r = 1, 2, \ldots, R$, with $|\underline{n}| := \sum_{r=1}^{R} n_r$ for all $\underline{n} = (n_1, \ldots, n_R) \in \mathbb{N}^R$.

Define $\pi(\underline{n}) := \lim_{t\to\infty} P(X_1(t) = n_1, \ldots, X_R(t) = n_R)$ to be the j.d.f. of the number customers of class $1, \ldots, R$ in the system in steady-state, where $X_r(t)$ is the number of customers of class r in the system at time t.

Define $\rho_r := \lambda_r/\mu_r$ for $r = 1, 2, \ldots, R$.

We have the following result:

Result 5.2 (Queue-length j.d.f. in a multiclass PS queue) *If* $\sum_{r=1}^{R} \rho_r < 1$ *(stability condition) then*

$$\pi(\underline{n}) = \frac{|\underline{n}|!}{G} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \tag{112}$$

for all $\underline{n} = (n_1, \ldots, n_R) \in \mathbb{N}^R$*, where*

$$G = \sum_{\underline{n}\in\mathbb{N}^R} |\underline{n}|! \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \tag{113}$$

$\square$

Proof. Since the arrival processes are Poisson and the service times are exponential it is seen that $((X_1(t),\ldots,X_R(t)),\ t \geq 0)$ is an irreducible C.-M.C. with state-space $\mathbb{N}^R$. Therefore, it suffices to check that (112) satisfies the balance equations.

The balance equations are:

$$\pi(\underline{n}) \sum_{r=1}^{R} \left(\mu_r \frac{n_r}{|\underline{n}|} + \lambda_r\right) = \sum_{r=1}^{R} \pi(\underline{n} - \underline{e}_r)\, \lambda_r\, \mathbf{1}(n_r > 0) + \sum_{r=1}^{R} \pi(\underline{n} + \underline{e}_r)\, \mu_r \frac{n_r + 1}{|\underline{n} + \underline{e}_r|} \tag{114}$$

for all $\underline{n} \in \mathbb{N}^R$.

Introducing (112) (after substituting $\underline{n}$ for $\underline{n} + \underline{e}_r$) in the second term in the right-hand side of (114) yields

$$\begin{aligned} \sum_{r=1}^{R} \pi(\underline{n} + \underline{e}_r)\, \mu_r \frac{n_r + 1}{|\underline{n} + \underline{e}_r|} &= \frac{1}{G} \sum_{r=1}^{R} |\underline{n} + \underline{e}_r|! \left(\prod_{s=1, s\neq r}^{R} \frac{\rho_s^{n_s}}{n_s!} \right) \frac{\rho_r^{n_r+1}}{(n_r+1)!} \mu_r \frac{n_r + 1}{|\underline{n} + \underline{e}_r|} \\ &= \pi(\underline{n}) \sum_{r=1}^{R} \frac{|\underline{n} + \underline{e}_r|!}{|\underline{n}|!} \frac{\lambda_r}{n_r + 1} \frac{n_r + 1}{|\underline{n} + \underline{e}_r|} \\ &= \pi(\underline{n}) \sum_{r=1}^{R} \lambda_r. \end{aligned} \tag{115}$$

Similarly, we get that the first term in the right-hand side of (114) satisfies

$$\sum_{r=1}^{R} \pi(\underline{n} - \underline{e}_r)\, \lambda_r\, \mathbf{1}(n_r > 0) = \pi(\underline{n}) \sum_{r=1}^{R} \mu_r \frac{n_r}{|\underline{n}|!}. \tag{116}$$

Adding now the terms in the right-hand sides of (115) and (116) gives us the left-hand side of (114), which concludes the proof.

$\blacksquare$

The remarkable result about the PS queue is the following: the j.d.f. $\pi(\underline{n})$ of the number of customers of class $1, 2, \ldots, R$ is given by (112) for <u>any</u> service time distribution of the customers of class $1, 2, \ldots, R$! In other words, (112) is <u>insensitive</u> to the service time distributions. The proof is omitted.

In particular, if the service time of customers of class r is constant and equals to S_r for $r = 1, 2, \ldots, R$, then

$$\pi(\underline{n}) = \frac{|\underline{n}|!}{G} \prod_{r=1}^{R} \frac{(\lambda_r\, S_r)^{n_r}}{n_r!}$$

for all $\underline{n} \in \mathrm{N}^R$, provided that $\lambda_r < 1/S_r$ for $r = 1, 2, \ldots, R$.

5.3 The Multiclass LCFS Queue

There are R classes of customers. Customers of class r enter the network according to a Poisson process with rate λ_r and require an exponential amount of service time with parameter μ_r. The K Poisson processes and service time processes are all assumed to be mutually independent processes.

There is a single server equipped with an infinite buffer. The service discipline is LCFS.

Define $\pi(\underline{n}) := \lim_{t\to\infty} P(X_1(t) = n_1, \ldots, X_R(t) = n_R)$ to be the j.d.f. of the number customers of class $1, \ldots, R$ in the system in steady-state, where $X_r(t)$ is the number of customers of class r in the system at time t.

The following result holds:

Result 5.3 (Queue-length j.d.f. in a multiclass LCFS queue) *If* $\sum_{r=1}^{R} \rho_r < 1$ *(stability condition) then*

$$\pi(\underline{n}) = \frac{|\underline{n}|!}{G} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \tag{117}$$

for all $\underline{n} = (n_1, \ldots, n_R) \in \mathrm{N}^R$, *where*

$$G = \sum_{\underline{n} \in \mathrm{N}^R} |\underline{n}|! \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \tag{118}$$

$\Box$

Observe that the j.d.f. $\pi(\underline{n})$ is the same as in the PS queue!

Proof. Here, $((X_1(t), \ldots, X_R(t)),\ t \geq 0)$ is <u>not</u> a C.-M.C. (can you see why?). A C.-M.C. for this queue is given $((I_1(t), \ldots, I_{N(t)}),\ t \geq 0)$ where $I_j(t) \in \{1, 2, \ldots, R\}$ is the class of the customer in the j-th position in the waiting room at time t and $N(t)$ is the total number of customers in the queue at time t.

The balance equations for this C.-M.C. are:

$$\pi'(r_1, \ldots, r_{n-1}, r_n) \left(\mu_{r_n} + \sum_{r=1}^{R} \lambda_r\right) = \lambda_{r_n} \pi'(r_1, \ldots, r_{n-1}) + \sum_{r=1}^{R} \pi'(r_1, \ldots, r_n, r)\, \mu_r \tag{119}$$

for all $(r_1, \ldots, r_{n-1}, r_n) \in \{1, 2, \ldots, R\}^n$, $n = 1, 2, \ldots$, and

$$\pi'(0) \sum_{r=1}^{R} \lambda_r = \sum_{r=1}^{R} \pi'(r)\, \mu_r \tag{120}$$

where $\pi'(0)$ is the probability that the system is empty.

It is straightforward to check that

$$\begin{aligned} \pi'(r_1, \ldots, r_n) &= \frac{1}{G} \prod_{i=1}^{n} \rho_{r_i} \quad \text{for } n = 1, 2, \ldots \\ \pi'(0) &= \frac{1}{G} \end{aligned} \tag{121}$$

satisfies the balance equations, where G is the normalizing constant.

Recall that $|\underline{n}| = \sum_{i=1}^{n} n_i$ for all $\underline{n} = (n_1, \ldots, n_R)$. For every fixed vector $\underline{n} = (n_1, \ldots, n_R) \in \mathbb{N}^R$, let $S(n_1, \ldots, n_R)$ denote the set of all vectors in $\{1, 2, \ldots, R\}^{|\underline{n}|}$ that have exactly n_1 components equal to 1, n_2 components equal to 2, ..., and n_R components equal to R.

Clearly, we have

$$\pi(\underline{n}) = \sum_{(i_1, \ldots, i_{|\underline{n}|}) \in S(n_1, \ldots, n_R)} \pi'(i_1, \ldots, i_{|\underline{n}|}) \tag{122}$$

for all $\underline{n} \in \mathbb{N}^R$, $\underline{n} \neq 0$, and $\pi(0, \ldots, 0) = \pi'(0) = 1/G$. From (121) and (122) we readily get (117). ∎

Again, we have the remarkable result that the product-form result (117) for the LCFS queue is insensitive to the service time distribution. The proof omitted.

5.4 The Multiclass IS Server Queue

There is an infinite number of servers so that a new customer enters directly a server. This queue is used to model delay phenomena in communication networks, for instance.

We keep the notation introduced in the previous sections. In particular, $\pi(\underline{n})$ is the j.d.f. that the number of customers of class $1, \ldots, R$ in steady-state (or, equivalently, the number of busy servers) is $n_1, \ldots, n_R$.

We have the following result:

Result 5.4 (Queue-length j.d.f. in a multiclass IS queue) *For all* $\underline{n} \in \mathbb{N}^R$,

$$\pi(\underline{n}) = \frac{1}{G} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \tag{123}$$

where

$$G = \sum_{\underline{n} \in \mathbb{N}^R} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \tag{124}$$

□

Observe that $\pi(\underline{n})$ is not equal to the corresponding quantity for PS (resp. LCFS) queues.

Proof. The process $((X_1(t), \ldots, X_R(t))\, t \geq 0)$ is a C.-M.C. The balance equations are:

$$\pi(\underline{n}) \sum_{r=1}^{R} (\lambda_r + \mu_r\, n_r) = \sum_{r=1}^{K} \pi(\underline{n} - \underline{e}_r)\, \lambda_r + \sum_{r=1}^{K} \pi(\underline{n} + \underline{e}_r)\, \mu_r\, (n_r + 1) \tag{125}$$

for all $\underline{n} = (n_1, \ldots, n_R) \in \mathbb{N}^R$.

It is easily checked that (123) satisfies the balance equations (125), which completes the proof.

■

Note that (123) holds for <u>any</u> values of the parameters $(\rho_r)_r$, or, equivalently, that an IS queue is always stable.

Again the product-form result (123) is insensitive to the service time distributions (proof omitted).

5.5 BCMP etworks

We now come to one of the main results of queueing network theory. Because of this result modeling and performance evaluation became popular in the late seventies and many queueing softwares based on the BCMP theorem became available at this time [7] (QNAP2 (INRIA-BULL), PAW (AT&T), PANACEA (IBM), etc.). Since then, queueing softwares have been continuously improved in the sense that they contain more and more analytical (e.g., larger class of product-form queueing models) and simulation tools (e.g., animated,

[7] This list is not at all exhaustive!

monitored, and controlled simulations), and are more user-friendly (e.g., graphical interfaces). Most queueing softwares are essentially simulation oriented (PAW, for instance); a few are hybrid (QNAP, PANACEA), meaning that they can handle both analytical models — queueing networks for which the "solution" is known explicitely, like open product-form queueing networks, or can be obtained through numerical procedures, like closed product-form queueing networks and closed markovian queueing networks — and simulation models. Simulation is more time and memory consuming but this is sometimes the only way to get (reasonably) accurate results. However, for large (several hundreds of nodes or more) non-product queueing networks both analytical and simulation models are not feasible in general, and one must then resort to approximation techniques. Approximation and simulation techniques will be discussed later on in this course.

We now state the so-called BCMP Theorem. It was obtained by F. Baskett, K. M. Chandy, R. R. Muntz and F. G. Palacios and their paper was published in 1975 ("Open, Closed, and Mixed Networks of Queues with Different Classes of Customers", *Journal of the Association for the Computing Machinery*, Vol. 22. No. 2, April 1975, pp. 248-260). This paper reads well and contain several simple examples that illustrate the main result.

A few words about the terminology.

We say that a node is of type FCFS (resp. PS, LCFS, IS) if it is a FCFS M/M/c queue (resp. PS, LCFS, IS queue, as defined above).

If node i is of type FCFS (we shall write $i \in \{FCFS\}$) then $\rho_{ir} := \lambda_{ir}/\mu_i$ for $r = 1, 2, \ldots, R$. As usual, μ_i is the parameter of the exponential service times in node i.

If node i is of type PS, LCFS, or IS (we shall write $i \in \{PS, LCFS, IS\}$) then $\rho_{ir} := \lambda_{ir}/\mu_{ir}$ for $r = 1, 2, \ldots, R$. Here, $1/\mu_{ir}$ is the mean service time for customers of type r in node i. For nodes of type PS, LCFS, or IS the service time distribution is arbitrary.

Recall that λ_{ir} is the solution to the traffic equations in Result 5.1. More precisely, if (i, r) belongs to an open chain then λ_{ir} is a solution to the first set of traffic equations in Result 5.1; otherwise, it is a solution to the second set of traffic equations.

Also recall the definition of $\alpha_i(j)$ for $i \in \{FCFS\}$ as well as the definition of $\gamma(j)$.

We have the following result:

Result 5.5 (BCMP Theorem) *For a BCMP network with K nodes and R classes of customers, which is open, closed or mixed in which each node is of type FCFS, PS, LCFS,* *or IS, the equilibrium state probabilities are given by*

$$\pi(\underline{n}) = \frac{d(\underline{n})}{G} \prod_{i=1}^{K} f_i(\underline{n}_i) \tag{126}$$

Formula (126) holds for any state $\underline{n} = (\underline{n}_1, \ldots, \underline{n}_K)$ *in the state-space* $\mathcal{S}$ *(that depends on the network under consideration) with* $\underline{n}_i = (n_{i1}, \ldots, n_{iR})$*, where* n_{ir} *is the number of customers of class* r *in node* i*. Moreover (with* $|\underline{n}_i| = \sum_{r=1}^{R} n_{ir}$ *for* $i = 1, 2, \ldots, K$*),*

- *if node* i *is of type FCFS, then*

$$f_i(\underline{n}_i) = |\underline{n}_i|! \prod_{j=1}^{|\underline{n}_i|} \frac{1}{\alpha_i(j)} \prod_{r=1}^{R} \frac{\rho_{ir}^{n_{ir}}}{n_{ir}!}; \tag{127}$$

- *if node* i *is of type PS or LCFS, then*

$$f_i(\underline{n}_i) = |\underline{n}_i|! \prod_{r=1}^{R} \frac{\rho_{ir}^{n_{ir}}}{n_{ir}!}; \tag{128}$$

- *if node* i *is of type IS, then*

$$f_i(\underline{n}_i) = \prod_{r=1}^{R} \frac{\rho_{ir}^{n_{ir}}}{n_{ir}!}. \tag{129}$$

In (126), $G < \infty$ *is the normalizing constant chosen such that* $\sum_{\underline{n} \in \mathcal{S}} \pi(\underline{n}) = 1$*,* $d(\underline{n}) = \prod_{j=0}^{M(\underline{n})-1} \gamma(j)$ *if the arrivals in the system depend on the total number of customers* $M(\underline{n}) = \sum_{i=1}^{K} |\underline{n}_i|$ *when the system is in state* $\underline{n}$*, and* $d(\underline{n}) = 1$ *if the network is closed.*

□

It is worth observing that the convergence of the series $\sum_{\underline{n} \in \mathcal{S}} \pi(\underline{n})$ imposes conditions on the parameters of the model, referred to as *stability conditions*.

The proof of Result 5.5 consists in writing down the balance equations for an appropriate C.-M.C. (can you see which one?), guessing a product-form solution, checking that this solution satisfies the balance equations, and finally aggregating the states in this solution (has done in Section 5.3) to derive (126).

Observe that (126) is a product-form result: node i behaves as if it was an isolated FCFS (resp. PS, LCFS, IS) node with input rates $(\lambda_{ir})_{r=1}^{R}$ if $i \in \{FCFS\}$ (resp. $i \in \{PS\}$, $i \in \{LCFS\}$, $i \in \{IS\}$).

Further generalizations of this result are possible: state-dependent routing probabilities, arrivals that depend on the number of customers in the subchain they belong to, more detailed state-space that gives the class of the customer in any position in any queue, etc. The last two extensions can be found in the BCMP paper whose reference has been given earlier.

Obtaining this result was not a trivial task; the consolation is that it is easy to use (which, overall, is still better than the contrary!). Indeed, the only thing that has to be done is to solve the traffic equations to determine the arrival rates $(\lambda_{ir})_{i,r}$, and to compute the normalizing constant G. For the latter computation, an extended convolution algorithm exists as well as many others. When you use a (hybrid) queueing software, these calculations are of course performed by the software. What you only need to do is to enter the topology of the network, that is, K the number of nodes and their type, R the number of classes, $[p_{i,r;j,s}]$ the matrix of routing probabilities, and to enter the values of the external arrival and service rates, namely, $(\lambda^0_{ir}\,\gamma(j), j = 0,1,2\ldots)_{i,r}$ the external arrival rates, $(\mu_i\alpha_i(j),\, j = 0,1,2,\ldots)_i$ the service rates for nodes of type $FCFS$, and $(\mu_{ir})_{i,r}$ the service rates of customers of class $r = 1,2,\ldots,R$ visiting nodes of type $\{PS, LCFS, IS\}$.

A further simplification in the BCMP theorem is possible if the network is open and if the arrivals do not depend upon the state of the system.

Introduce $R_i = \{\text{class r customers that may require service at node i}\}$. Define $\rho_i = \sum_{r\in R_i} \rho_{ir}$ (ρ_{ir} has been defined earlier in this section). Let $\pi_i(n)$ be the probability that node i contains n customers.

We have the following result:

Result 5.6 (Isolated node) *For all $n \geq 0$,*

$$\pi_i(n) = (1-\rho_i)\,\rho_i^n$$

if $i \in \{FCFS, PS, LCFS\}$, and

$$\pi_i(n) = e^{-\rho_i}\,\frac{\rho_i^n}{n!}$$

if $i \in \{IS\}$.

□

6 Case Studies

In this section we shall examine two systems and determine their performance measures by building simple models. The first study will require the application of Jackson's result. The second one will require the application of the BCMP Theorem.

6.1 Message-Switching Network

We consider a M-channel, N-node message-switching communication network. The M communication channels are supposed to be noiseless, perfectly reliable, and to have a capacity denoted by C_i bits per second for channel i. The N nodes refer to the message-switching centers which are also considered to be perfectly reliable, and in which all of the message-switching functions take place, including such things as message reassembly, routing, buffering, acknowledging, and so on.

We will assume that the nodal processing time at each node is constant and equal to K. In addition, there are of course the channel queueing delays and the transmission delays.

We will assume that the (external) traffic entering the network forms a Poisson process with rate γ_{jk} (messages per second) for those messages originating at node j and destined for node k. Let

$$\gamma = \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_{jk}$$

be the total external traffic rate.

All messages are assumed to have lengths that are drawn independently from an exponential distribution with mean $1/\mu$ (bits). In order to accomodate these messages, we assume that all nodes in the network have unlimited storage capacity.

We will further assume for the sake of simplicity that there exists a unique path through the network for a given origin-destination pair (this assumption is not essential).

In high-speed networks spanning large geographical areas it may be important to include the propagation time P_i which is the time required for a single bit to travel along channel i, $i = 1, 2, \ldots, M$. If the channel has a length of l_i km then $P_i = l_i/v$ where v is the speed of light. Thus, if a message has a length b bits then the time it occupies channel i will be

$$P_i + \frac{b}{C_i} \text{ sec.} \tag{130}$$

Note that the randomness in the service time comes not from the server (the channel) but from the customer (the message) in that the message length is a random variable $\tilde{b}$.

At first glance, the network we have defined so far is similar to an open Jackson queueing network. However, this is not true since in the present network the service times at different nodes may not be independent unlike in a Jackson network. This follows from the fact that the service time of a message is directly related to its length (see (130)) and from the fact that a message will in general visit several nodes along its route in the network. Except for this difficulty, one could apply Jackson's result to this network immediately. We will continue this discussion later on.

Since each channel in the network is considered to be a separate server, we adopt the notation λ_i as the average number of messages per second which travel over channel i. As with the external traffic we define the total traffic rate within the network by

$$\lambda = \sum_{i=1}^{M} \lambda_i. \tag{131}$$

The message delay is the total time that a message spends in the network. Of most interest is the average message delay T and we take this as our basic measure performance.

We shall say that a message is of type (j, k) if the origin of the message is node j and its destination is node k.

Define the quantity

$$Z_{jk} = \mathrm{E}\left[\text{message delay for a message of type } (j,k)\right].$$

It should be clear that

$$T = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\gamma_{jk}}{\gamma} Z_{jk} \tag{132}$$

since the fraction γ_{jk}/γ of the total traffic will suffer the delay Z_{jk} on the average. Note that (132) represents a decomposition of the network on the basis of origin-destination pairs.

Our goal is to solve for T. Let us denote by a_{jk} the path taken by messages that originate at node j and that are destined for node k. We say that the channel i (of capacity C_i) is contained in a_{jk} if messages of class (j, k) traverse channel i; in such case we use the notation $C_i \in a_{jk}$.

It is clear therefore that the average rate of message flow λ_i on channel i must be equal to the sum of the average message flows of all paths that traverse this channel, that is,

$$\lambda_i = \sum_{\{(j,k)\,:\,C_i \in a_{jk}\}} \gamma_{jk}. \tag{133}$$

Moreover, we recognize that Z_{jk} is merely the sum of the average delays encountered by a message in using the various channels along the path a_{jk}. Therefore,

$$Z_{jk} = \sum_{\{i\,:\,C_i \in a_{jk}\}} T_i \tag{134}$$

where T_i is expected time spent waiting for <u>and</u> using channel i.

From (132) and (134) we have

$$T = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\gamma_{jk}}{\gamma} \sum_{\{i\,:\,C_i \in a_{jk}\}} T_i.$$

We now exchange the order of summations and observe that the condition on i becomes the corresponding condition on the pair (j, k) as it is usual when interchanging summations; this yields

$$T = \sum_{i=1}^{M} \frac{T_i}{\gamma} \sum_{\{(j,k)\,:\,C_i \in a_{jk}\}} \gamma_{jk}.$$

Using (133) we finally have

$$T = \sum_{i=1}^{M} \frac{\lambda_i}{\gamma} T_i. \tag{135}$$

We have now <u>decomposed</u> the average message delay into its single-channel components, namely the delay T_i.

We are now left with computing T_i. L. Kleinrock discussed the problem of modeling this network as a Jackson network despite the fact that the (exponential) service times at different nodes are not independent. Through numerous simulation results he got to the conclusion that if messages entering a given channel depart on several channels or if messages leaving the node on a given channel had entered from distinct channels, then the following assumption is reasonable:

Independence Assumption: each time a message is received at a node within the network, a <u>new length</u>, b, is chosen independently with the density function $\mu \exp(-\mu x)$ for $x \geq 0$.

From now on we shall assume that this assumption holds. Assume first that $P_i = 0$ (no propagation delay) and $K = 0$ (no nodal processing time). Under this independence assumption between the service times at different nodes, we know from the Jackson result that channel i is now representable as an M/M/1 queue with arrival rate λ_i and service rate μC_i.

If the stability condition $\lambda_i < \mu\, C_i$ for $i = 1, 2, \ldots, M$ holds, then the solution for T_i is given by

$$T_i = \frac{1}{\mu\, C_i - \lambda_i} \qquad \forall i = 1, 2, \ldots, M.$$

If $P_i \neq 0$ and $K > 0$ then

$$T_i = \frac{1}{\mu\, C_i - \lambda_i} + P_i + K$$

and so, from (135) we obtain

$$T = \sum_{i=1}^{M} \frac{\lambda_i}{\gamma} \left(\frac{1}{\mu\, C_i - \lambda_i} + P_i + K \right). \tag{136}$$

We now want to solve the following optimization problem called the

Capacity Assignment Problem:

Given:	Flows $\{\lambda_i\}_{i=1}^{M}$ and network topology
Minimize:	T
With respect to:	$\{C_i\}_{i=1}^{M}$
Under the constraint:	$D = \sum_{i=1}^{M} C_i$.

To solve this problem we will use the Lagrange multipliers. Let us recall what Lagrange multipliers are.

Let $f(x_1, \ldots, x_n)$ be an expression whose extreme values are sought when the variables are restricted by a certain number of side conditions, say $g_1(x_1, \ldots, x_n) = 0, \ldots, g_m(x_1, \ldots, x_n) = 0$. We then form the linear combination

$$h(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) + \sum_{k=1}^{m} \beta_k\; g_k(x_1, \ldots, x_n)$$

where $\beta_1, \ldots, \beta_m$ are m constants, called the Lagrange multipliers. We then differentiate h with respect to each x_i and consider the system of $n + m$ equations:

$$\begin{aligned} \frac{\partial}{\partial x_i} h(x_1, \ldots, x_n) &= 0 \qquad i = 1, 2, \ldots, n \\ g_k(x_1, \ldots, x_n) &= 0 \qquad k = 1, 2, \ldots, m. \end{aligned}$$

Lagrange discovered that if the point $(x_1, \ldots, x_n)$ is a solution of the extremum problem then it will also satisfy this system of $n+m$ equations. In practice, one attemps to solve this system for the $n+m$ "unknown" $x_1, \ldots, x_n$, $\beta_1, \ldots, \beta_m$. The points $(x_1, \ldots, x_n)$ so obtained must then be tested to determine whether they yield a maximum, a minimum, or neither.

We now come back to our optimization problem. We shall assume without loss of generality that $P_i = 0$ for $i = 1, 2, \ldots, M$ and $K = 0$.

Setting $\rho_i = \lambda_i/\mu$ we see from (136) that

$$T = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\rho_i}{C_i - \rho_i}$$

The stability condition now reads $\rho_i < C_i$ for $i = 1, 2, \ldots, M$. Observe, in particular, that $\sum_{i=1}^{M} \rho_i < \sum_{i=1}^{M} C_i = D$ under the stability condition.

Define

$$f(C_1, \ldots, C_M) = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\rho_i}{C_i - \rho_i} + \beta \left(\sum_{i=1}^{M} C_i - D \right).$$

According to Lagrange, we must solve the equation $\partial f(C_1, \ldots, C_n)/\partial C_i = 0$ for every $i = 1, 2, \ldots, N$. We obtain

$$C_i = \rho_i + \sqrt{\frac{\rho_i}{\gamma\beta}} \qquad i = 1, 2, \ldots, M. \tag{137}$$

It remains to determine the last unknown β. From the constraint $\sum_{i=1}^{M} C_i = D$ we see from (137) that

$$\frac{1}{\sqrt{\gamma\beta}} = \frac{D - \sum_{i=1}^{M} \rho_i}{\sum_{i=1}^{M} \sqrt{\rho_i}}.$$

Introducing this value of $1/\sqrt{\gamma\beta}$ in (137) finally yields

$$C_i = \rho_i + \frac{\sqrt{\rho_i}}{\sum_{i=1}^{M} \sqrt{\rho_i}} \left(D - \sum_{i=1}^{M} \rho_i \right) \qquad i = 1, 2, \ldots, M. \tag{138}$$

We observe that this assignment allocates capacity such that each channel receives at least ρ_i (which is the minimum under the stability condition) and then allocates some additional capacity to each.

6.2 Buffer Overflow in a Store-and-Forward Network

We consider a communication network with N nodes where packets traverse the network in a store-and-forward fashion. After a node transmits a packet to another node (nodes are called IMP's in the networking terminology, where the acronym IMP stands for Interface Message Processor), it waits for a positive acknowledgment (ACK) from the receiving node before relinquishing responsability for the packet (i.e., upon reception of an ACK the copy of the acknowledged packet that had been created upon arrival of the packet is destroyed). If a node does not receive an ACK within a given interval of time (timeout), the packet is retransmitted after a suitable interval of time. Upon receipt of a packet, the receiving node checks for transmission errors (e.g., parity errors) and determines whether there is enough room to store the packet. If there is room for the packet and if it is error-free, the receiving node issues an ACK to the sending node. In this fashion, a packet travels over the links between the source node and the destination node.

In the absence of congestion control mecanisms it is well-know that the throughput of the network as function of the offered load has a knee shape: it linearly increases up to a certain value λ_0 of the offered load and then rapidly decreases to 0. We say that the network is congested when the offered load is larger than λ_0. The congestion mode yields high response times and unstability in the network, and is therefore highly undesirable. The congestion mode can be avoided by controlling the access to the network.

A commonly used mecanism is a mecanism where at each IMP the number of packets waiting for an ACK can not exceed a given value; new packets arriving at the node when this value is reached are rejected (i.e., no ACK is sent).

We shall assume that node i cannot contain more than K_i non-acknowledged packets. Even if we take Markovian assumptions for the arrival traffic (Poisson assumption) and for the transmission times (exponential transmission times), this network cannot be modeled as a product-form queueing network (Jackson network, BCMP network) because of the blocking phenomena.

We are going to present an approximation method that will enable us to determine the buffer overflow at each node in the network.

We first model and study a single node in the network, say node i.

Node i is composed of

(1) a central processor denoted as station 0;

(2) O_i output channels denoted as stations $1, 2, \ldots, O_i$;

(3) O_i timeout boxes denoted as stations $1', 2' \ldots, O_i'$;

(4) O_i ACK boxes denoted as stations $1'', 2'' \ldots, O_i''$.

Upon arrival of a new packet at node i the central processor checks for errors and determines whether there is room for this packet in the node; depending upon the outcome of these tests an ACK is sent to the transmitting node or the packet is discarded at once (and no ACK is sent).

Assume that an ACK is sent. Then, the processor determines the destination of the accepted packet and places it on the appropriate output channel buffer. Packets are then transmitted over the appropriate output line. Let $\{0, 1, 2, \ldots, O(i)\}$ denote the set of lines out of node i, where line 0 represents the exit from the network.

At the receiving node, the process is repeated. The transmission-retransmission process is modeled by the timeout and the ACK boxes. More precisely, it is assumed that with probability q_{ij} the attempted transmission over output channel $j = 1, 2, \ldots, O_i$ fails, either through blocking or through channel error. We model this event as having the packet enter the timeout box where it resides for a random interval of time. The probability of successful transmission over output channel j (i.e., $1 - q_{ij}$) is modeled as having the packet enter the ACK box for a random period of time. The time in the output channel represents the transmission time to the destination node. The residence times in ACK boxes represent such quantities as propagation and processing times at the receiver.

Let E_{ij} be the probability that a packet sent over the channel (i, j) has one or more bits in error. Let B_i be the blocking probability at station i. Assuming that nodal blocking and channel errors are independent events, the steady-state probability $1 - q_{ij}$ of success for a packet transmitted over channel (i, j) is given by

$$1 - q_{ij} = (1 - E_{ij})\,(1 - B_j). \tag{139}$$

With probability $(1 - q_{ij})(q_{ij})^n$ a packet is retransmitted exactly n times over channel (i, j) before success. Hence, the mean number of transmissions for a packet over channel (i, j) is $1/(1 - q_{ij})$.

Let us prove this result. We have that q_{ij} is the probability that a transmission over channel (i, j) fails. The probability of having n <u>retransmissions</u> is $(1 - q_{ij})\, q_{ij}^n$ and the mean number of <u>retransmissions</u> for a packet is

$$(1 - q_{ij}) \sum_{n \geq 0} n\, q_{ij}^n = \frac{q_{ij}}{1 - q_{ij}}.$$

(Hint: use the fact that $\sum_{n\geq 0} x^n = 1/(1-x)$ for $|x|<1$ and differentiate this series to get $\sum_{n\geq 1} n\,x^{n-1} = 1/(1-x)^2$.)

Therefore, the mean number of transmissions per packet is $1 + q_{ij}/(1-q_{ij}) = 1/(1-q_{ij})$.

Statistical Assumptions:

We assume that the packets arrive at node i according to a Poisson process with rate γ_i. With the probability p_{ij}, $j = 1, 2, \ldots, O_i$, a packet is destined to output j.

The packets are assumed to have exponential length. Therefore, each output channel is modeled as a FCFS queue with service rate μ_{ij}, $j = 1, 2, \ldots, O_i$. The central processor is also modeled as a FCFS queue with service rate μ_i. Each timeout box and each ACK box is modeled as an IS queue. We denote by $1/\tau_{ij}$ and $1/\alpha_{ij}$ the mean service time in timeout box and in ACK box j, respectively, for $j = 1, 2, \ldots, O_i$.

Under these assumptions node i is actually an open BCMP network with $3\,O_i + 1$ stations including $O_i + 1$ FCFS stations and $2O_i$ IS stations, and depending arrival rate $\gamma_i(\underline{n})$ given by

$$\gamma_i(\underline{n}) = \begin{cases} \gamma_i & \text{if } |\underline{n}| < K_i \\ 0 & \text{otherwise} \end{cases}$$

where $\underline{n} = (n_0, n_1, \ldots, n_{O_i}, n'_1, \ldots, n'_{O_i}, n''_1, \ldots, n''_{O_i})$ with $|\underline{n}| \leq K_i$. Here n_0, n_j, n'_j and n''_j are the number of customers in station 0, j, j' and j'', respectively, in state $\underline{n}$.

Let us solve for the traffic equations. Let λ_i, λ_{ij}, λ'_{ij} and λ''_{ij} be the arrival rate in station 0, j, j' and j'', respectively, for $j = 1, 2, \ldots, O_i$.

The traffic equations read

$$\begin{aligned} \lambda_i &= \gamma_i \\ \lambda_{ij} &= \lambda_i\, p_{ij} + \lambda'_{ij} \\ \lambda'_{ij} &= \lambda_{ij}\, q_{ij} \\ \lambda''_{ij} &= \lambda_{ij}\,(1 - q_{ij}) \end{aligned}$$

for $j = 1, 2, \ldots, O_i$.

We find

$$\lambda_{ij} = \frac{\gamma_i\, p_{ij}}{1 - q_{ij}}$$

$$\begin{aligned}
\lambda'_{ij} &= \frac{q_{ij}}{1-q_{ij}}\,\gamma_i\,p_{ij} \\
\lambda''_{ij} &= \gamma_i\,p_{ij}
\end{aligned}$$

for $j = 1, 2, \ldots, O_i$.

Let $\pi_i(\underline{n})$ be the stationary probability of being in state $\underline{n} \in \mathcal{S}_i$ where

$$\mathcal{S}_i = \{(n_0, n_1, \ldots, n_{0_i}, n'_1, \ldots, n'_{O_i}, n''_1, \ldots, n''_{O_i}) \;:\; |\underline{n}| \le K_i\}.$$

Here n_0, n_j, n'_j and n''_j are the number of customers in station 0, j, j' and j'', respectively, for $j = 1, 2, \ldots, O_i$, when node i is in state $\underline{n}$.

From the BCMP result we have:

$$\pi_i(\underline{n}) = \frac{1}{G_i}\left(\frac{\gamma_i}{\mu_i}\right)^{n_0} \prod_{j=1}^{O_i} \left\{ \left(\frac{\lambda_{ij}}{\mu_{ij}}\right)^{n_j} \frac{1}{n'_j!}\left(\frac{\lambda'_{ij}}{\tau_{ij}}\right)^{n'_j} \frac{1}{n''_j!}\left(\frac{\lambda''_{ij}}{\alpha_{ij}}\right)^{n''_j} \right\} \tag{140}$$

for all $\underline{n} \in \mathcal{S}_i$ and

$$\pi_i(\underline{n}) = 0 \quad \text{for } \underline{n} \notin \mathcal{S}_i,$$

where G_i is a normalizing constant chosen such that $\sum_{\underline{n}\in\mathcal{S}_i} \pi_n(\underline{n}) = 1$.

Let B_i be the blocking probability in node i, namely,

$$B_i = \sum_{\underline{n}\in\mathcal{S}_i, |\underline{n}|=K_i} \pi_i(\underline{n}). \tag{141}$$

Having found the probability of blocking for a node for a given amount of traffic into the node, we now consider the entire network. To properly interface the single node results, we invoke the conservation flows principle, namely, <u>the total flow out must equal the total flow in</u>.

Let Λ_i be the <u>effective</u> arrival rate at node i for $i = 1, 2, \ldots, N$. Under equilibrium conditions, we must have

$$\Lambda_i = \lambda_i^0 + \sum_{j=1}^{N} \Lambda_j\,p_{ji}$$

where λ_i^0 is the input rate at node i in packets per second of packets coming from outside the network.

Because of retransmissions, the total arrival rate γ_i at node i is actually larger than Λ_i. We have already seen that the mean number of transmissions for a packet over channel (i, j) is $1/(1 - q_{ij})$.

Hence, the total arrival rate γ_i at node i is given by

$$\begin{aligned}\gamma_i &= \frac{\lambda_i^0}{1-q_{0i}} + \sum_{j=1}^{N} \frac{\Lambda_j}{1-q_{ji}} p_{ji} \\ &= \left[\frac{\lambda_i^0}{1-E_{0i}} + \sum_{j=1}^{N} \frac{\Lambda_j}{1-E_{ji}} p_{ji}\right] \frac{1}{1-B_i} \end{aligned} \tag{142}$$

for $i = 1, 2, \ldots, N$.

By pluging the values of γ_i, $i = 1, 2, \ldots, N$, in (141) we obtain a set of N nonlinear equations which can be solved for the N blocking probabilities,

$$B_i = f_i(B_1, \ldots, B_N) \tag{143}$$

for $i = 1, 2, \ldots, N$.

This set of nonlinear equations can be solved by using the Newton-Raphson method. Define $B = (B_1, \ldots, B_N)$ and $f = (f_1, \ldots, f_N)$.

Starting with an initial value B^0 for B, the Newton-Raphson method successively generates a sequence of vectors $B^1, B^2, \ldots, B^k, \ldots$ that converges to a solution of (143).

More precisely,

$$B^{k+1} = B^k - [I - \nabla f(B^k)]^{-1} [B^k - f(B^k)]$$

where I is the N-by-N identity matrix, and where ∇f, the gradient of f with respect to B, is defined to be the following N-by-N matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial B_1} & \cdots & \frac{\partial f_1}{\partial B_N} \\ \frac{\partial f_2}{\partial B_1} & \cdots & \frac{\partial f_2}{\partial B_N} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_N}{\partial B_1} & \cdots & \frac{\partial f_N}{\partial B_N} \end{pmatrix}.$$

For a stopping condition, define the k-th estimation error estimate to be

$$\eta_k = \frac{\|f(B^k) - B^k\|}{\|f(B_k)\|}$$

where

$$\|x\| = \sqrt{\sum_{i=1}^{N} x_i^2}$$

for $x = (x_1, \ldots, x_N)$. B^k is accepted as a solution if η_k is smaller than some prespecified convergence tolerance.

7 Queueing Theory (continued)

7.1 The FCFS GI/GI/1 Queue

A GI/GI/1 queue is a single server queue with arbitrary, but independent, service times and interarrival times. More precisely, let s_n be the service time of the n-th customer, and let τ_n be the time between arrivals of customers n and $n+1$.

In a GI/GI/1 queue we assume that:

- A1 $(s_n)_n$ is a sequence of independent random variables with the common cumulative distribution function (c.d.f.) $G(x)$, namely, $P(s_n \leq x) = G(x)$ for all $n \geq 1$, $x \geq 0$;
- A2 $(s_n)_n$ is a sequence of independent random variables with the common c.d.f. $F(x)$, namely, $P(\tau_n \leq x) = F(x)$ for all $n \geq 1$, $x \geq 0$;
- A3 s_n and τ_m are independent r.v.'s for all $m \geq 1$, $n \geq 1$.

We shall assume that $F(x)$ and $G(x)$ are both differentiable in $[0, \infty)$ (i.e., for every $n \geq 1$, s_n and τ_n each have a density function).

In the following, we shall assume that the service discipline is FCFS. Let $\lambda = 1/E[\tau_n]$ and $\mu = 1/E[s_n]$ be the arrival rate and the service rate, respectively.

Let W_n be the waiting time in queue of the n-th customer.

The following so-called Lindley's equation holds:

$$W_{n+1} = \max(0, W_n + s_n - \tau_n) \quad \forall n \geq 1. \tag{144}$$

From now on, we shall assume without loss of generality that $W_1 = 0$, namely, the first customer enters an empty system.

It can be shown that the system is stable, namely, $\lim_{n\to\infty} P(W_n \leq x) = P(W \leq x)$ for all $x \geq$, where W is an almost surely finite r.v., if

$$\lambda < \mu \tag{145}$$

which should not be a surprising result. The proof of (145) is omitted (if $\lambda > \mu$ then the system is always unstable, that is, the queue is unbounded with probability one).

Let $W := \lim_{n\to\infty} W_n$ under the stability condition (145).

Define $\phi(\theta) = E[\exp(\theta(s_n - \tau_n))]$ the Laplace transform of the r.v. $s_n - \tau_n$. We shall assume that there exists $c > 0$ such that $\phi(c) < \infty$. In practice, namely for all interesting c.d.f.'s for the interarrival times and service times, this assumption is <u>always</u> satisfied.

Therefore, since $\phi(0) = 1$ and since

$$\phi'(0) = E[s_n - \tau_n] = (\lambda - \mu)/(\lambda\,\mu) < 0$$

from (145), we know that there exists $\theta > 0$ such that $\phi(\theta) < 1$.

Our goal is to show the following result:

Result 7.1 (Exponential bound for the GI/GI/1 queue) *Assume that* $\lambda < \mu$. *Let* $\theta > 0$ *be such that* $\phi(\theta) \leq 1$.

Then,

$$P(W_n \geq x) \leq e^{-\theta x} \quad \forall x > 0,\, n \geq 1 \tag{146}$$

and

$$P(W \geq x) \leq e^{-\theta x} \quad \forall x > 0. \tag{147}$$

□

The proof is remarkably simple for such a general queueing system and is due to J. F. C. Kingman ("Inequalities in the Theory of Queues", *Journal Royal Statistical Society*, Series B, Vol. 32, pp. 102-110, 1970).

Proof of Result 7.1: Let f be a non-increasing function on $(0,\infty)$ with $0 \leq f(x) \leq 1$, such that, for all $x > 0$,

$$\int_{-\infty}^{x} f(x-y)\,dH(y) + 1 - H(x) \leq f(x) \tag{148}$$

where $H(x)$ is the c.d.f. of the r.v. $s_n - \tau_n$, namely, $H(x) = P(s_n - \tau_n \leq x)$ for all $x \in (-\infty,\infty)$, and where $dH(x) := H'(x)\,dx$.

Let us show that

$$P(W_n \geq x) \leq f(x) \quad \forall x > 0,\, n \geq 1. \tag{149}$$

We use an induction argument.

The result is true for $n = 1$ since $W_1 = 0$. Therefore, $P(W_1 \geq x) = 0 \leq f(x)$ for all $x > 0$. Assume that $P(W_m \geq x) \leq f(x)$ for all $x > 0$, $m = 1, 2, \ldots, n$ and let us show that this is still true for $m = n + 1$.

We have, for all $x > 0$ (cf. (144)),

$$\begin{aligned} P(W_{n+1} \geq x) &= P(\max(0, W_n + s_n - \tau_n) \geq x) \\ &= P(W_n + s_n - \tau_n \geq x) \end{aligned}$$

since for any r.v. X, $P(\max(0,X) \geq x) = 1 - P(\max(0,X) < x) = 1 - P(0 < x, X < x) = 1 - P(X < x) = P(X \geq x)$ for all $x > 0$.

Thus, for all $x > 0$,

$$\begin{aligned} P(W_{n+1} \geq x) &= P(W_n + s_n - \tau_n \geq x) \\ &= \int_{-\infty}^{\infty} P(W_n \geq x - y \,|\, s_n - \tau_n = y)\, dH(y) \\ &= \int_{-\infty}^{\infty} P(W_n \geq x - y)\, dH(y) \quad \text{since } W_n \text{ is independent of } s_n \text{ and } \tau_n \\ &= \int_{-\infty}^{x} P(W_n \geq x - y)\, dH(y) + \int_{x}^{\infty} dH(y) \text{ since } P(W_n \geq u) = 1 \text{ for } u \leq 0 \\ &\leq \int_{-\infty}^{x} f(x - y)\, dH(y) + 1 - H(x) \quad \text{from the induction hypothesis} \\ &\leq f(x) \end{aligned}$$

from (148).

Letting now $n \to \infty$ in (149) gives

$$P(W \geq x) \leq f(x) \quad \forall x > 0. \tag{150}$$

Let us now show that the function $f(x) = \exp(-\theta x)$ satisfies (148) which will conclude the proof. We have

$$\begin{aligned} &\int_{-\infty}^{x} e^{\theta(y-x)}\, dH(y) + 1 - H(x) \\ &= e^{-\theta x} \int_{-\infty}^{\infty} e^{\theta y}\, dH(y) - \int_{x}^{\infty} e^{\theta(y-x)}\, dH(y) + 1 - H(x) \end{aligned}$$

$$= e^{-\theta x}\,\phi(\theta) - \int_x^\infty \left(e^{\theta(y-x)} - 1\right)\, dH(y)$$
$$\le e^{-\theta x}$$

since $\phi(\theta) \le 1$ by assumption and $\exp(\theta(y-x)) - 1 \ge 0$ for all $y \ge x$. ∎

Remark 7.1 (Independence of W_n and (s_n, τ_n)) In the proof of Result 7.1, we have used the fact that W_n is independent of the r.v.'s s_n and τ_n. This result comes from the fact that, for all $n \ge 2$, W_n is only a function of $s_1, \ldots, s_{n-1}, \tau_1, \ldots, \tau_{n-1}$. More precisely, it is straighforward to see from (144) and the initial condition $W_1 = 0$ that

$$W_n = \max\left(0, \max_{i=1,2,\ldots,n-1} \sum_{j=i}^{n-1} (s_i - \tau_i)\right) \quad \forall n \ge 2. \tag{151}$$

Therefore, W_n is independent of s_n and τ_n since s_n and τ_n are independent of $(s_i, \tau_i)_{i=1}^{n-1}$ from assumptions A1, A2 and A3.

Remark 7.2 ($(W_n)_n$ is a Markov chain) Let us show that $(W_n)_n$ is a discrete-time continuous-time Markov chain. For this, we must check that $P(W_{n+1} \le y \,|\, W_1 = x_1, \ldots, W_{n-1} = x_{n-1}, W_n = x)$ is only a function of x and y, for all $n \ge 1$, $x_1, \ldots, x_{n-1}, x, y$ in $[0, \infty)$.

We have from (144)

$$\begin{aligned}
&P(W_{n+1} \le y \,|\, W_1 = x_1, \ldots, W_{n-1} = x_{n-1}, W_n = x) \\
&\quad = \quad P(\max(0, x + s_n - \tau_n) \le y \,|\, W_1 = x_1, \ldots, W_{n-1} = x_{n-1}, W_n = x) \\
&\quad = \quad P(\max(0, x + s_n - \tau_n) \le y)
\end{aligned}$$

since we have shown in Remark 7.1 that s_n and τ_n are independent of W_j for $j = 1, 2, \ldots, n$.

The best exponential decay in (146) and (147) is the largest $\theta > 0$, denoted as θ^*, such that $\phi(\theta) \le 1$. One can show (difficult) that θ^* is actually the largest exponential decay, which means there does not exist θ such that $\phi(\theta) > 1$ and $\theta > \theta^*$.

From Result 7.1 we can easily derive an upper bound for $E[W_n]$ and $E[W]$.

Result 7.2 ((Upper bound for the transient and stationary mean waiting time) *Assume that $\lambda < \mu$. Then,*

$$E[W_n] \le \frac{1}{\theta^*} \quad \forall n \ge 2, \tag{152}$$

and

$$E[W] \leq \frac{1}{\theta^*}. \tag{153}$$

□

Proof. It is known that $E[X] = \int_0^\infty P(X > x)\, dx$ for any nonnegative r.v. X. From Result 7.1 and the above identity we readily get (152) and (153).

■

The bound on the stationary mean waiting time obtained in (153) can be improved using more sophisticated techniques.

7.2 Application: Effective Bandwidth in Multimedia Networks

In high-speed multimedia networks, admission control plays a major role. Because of the extreme burstiness of some real-time traffic (e.g. video), accepting a new session in a network close to congestion may be dramatic. On the other hand, rejecting to many users maybe very costly. Also, because of the high speeds involved, admission control mecanisms must be very fast in making the decision to accept/reject a new session.

On the other hand, an interesting feature about real-time traffic applications is that they are able to tolerate a small fraction of packets missing their deadline (e.g., approx. 1% for voice). Therefore, <u>bounds on the tail distribution</u> of quantities such as buffer occupancy and response times can be used by designers to size the network as well as to develop efficient admission control mecanisms.

Assume that the system may support K different types of sessions (e.g., voice, data, images). Assume that there are n_1 active sessions of type 1, ..., n_K active sessions of class K, upon arrival of a new session of type i. We would like to answer the following questions: should we accept or rejet this new session so that

Problem 1:

$$P(W \geq b) \leq q \tag{154}$$

Problem 2:

$$E[W] < \alpha \tag{155}$$

where $\alpha, b > 0$ and $q \in (0, 1)$ are prespecified numbers. Here, W is the stationary delay (i.e., waiting time in queue).

For each problem, the decision criterion has to be simple enough so that decisions to admit/reject new sessions can be made very rapidly, and easily implementable.

We will first consider the case when the inputs are independent Poisson processes.

7.2.1 Effective Bandwidth for Poisson Inputs

Consider a M/G/1 queue with N (non-necessarily distinct) classes of customers. Customers of class k are generated according to a Poisson process with rate λ_k; let $G_k(x)$ be the c.d.f. of their service time and let $1/\mu_k$ be their mean service time. We assume that the arrival time and service time processes are all mutually independent processes.

We first solve the optimization problem (154).

The form of the criterion (154) strongly suggests the use of Result 7.1. For this, we first need to place this multiclass M/G/1 queue in the setting of Section 1, and then to compute $\phi(\theta)$.

First, let us determine the c.d.f. of the time between two consecutive arrivals. Because the superposition of N independent Poisson processes with rates $\lambda_1, \ldots, \lambda_N$ is a Poisson process with rate

$$\lambda := \sum_{i=1}^{N} \lambda_i \tag{156}$$

we have that $P(\tau_n \leq x) = 1 - \exp(-\lambda x)$ for all $x \geq 0$.

Let us now focus on the service times. With the probability λ_k/λ the n-th customer will be a customer of class k. Let us prove this statement. Let X_k be the time that elapses between the arrival of the n-th customer and the first arrival of a customer of class k. Since the arrival process of each class is Poisson, and therefore memoryless, we know that X_k is distributed according to an exponential r.v. with rate λ_k, and that $X_1, \ldots, X_N$ are independent r.v.'s.

Therefore,

$$\begin{aligned}
P((n+1)\text{-st customer is of class } k) &= P(X_k < \min_{i \neq k} X_i) \\
&= \int_0^\infty P(x < \min_{i \neq k} X_i)\, \lambda_k\, e^{-\lambda_k x}\, dx \\
&= \int_0^\infty \prod_{i \neq k} P(x < X_i)\, \lambda_k\, e^{-\lambda_k x}\, dx \\
&= \lambda_k \int_0^\infty \prod_{i=1}^{N} e^{-\lambda_i x}\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_k \int_0^\infty e^{-\lambda x}\, dx \\
&= \frac{\lambda_k}{\lambda}.
\end{aligned}
$$

Let us now determine the c.d.f. $G(x)$ of the service time of an arbitrary customer. We have

$$
\begin{aligned}
G(x) &= P(\text{service time new customer } \leq x) \\
&= \sum_{k=1}^{N} P\left(\text{service time new customer } \leq x \,\text{and new customer of type } k\right) \quad (157) \\
&= \sum_{k=1}^{N} P\left(\text{service time new customer } \leq x \,|\, \text{new customer of type } k\right) \frac{\lambda_k}{\lambda} \quad (158)
\end{aligned}
$$

Here, (157) and (158) come from the law of total probability and from Bayes' formula, respectively.

Thus,

$$
G(x) = \sum_{k=1}^{N} \frac{\lambda_k}{\lambda}\, G_k(x). \tag{159}
$$

In particular, the mean service time $1/\mu$ of this multiclass M/G/1 queue is given by

$$
\begin{aligned}
\frac{1}{\mu} &= \int_0^\infty x\, dG(x) \\
&= \sum_{k=1}^{N} \frac{\lambda_k}{\lambda} \int_0^\infty x\, dG_k(x) \\
&= \sum_{k=1}^{N} \frac{\rho_k}{\lambda} \qquad (160)
\end{aligned}
$$

with $\rho_k := \lambda_k/\mu_k$.

In other words, we have "reduced" this multiclass M/G/1 queue to a G/G/1 queue where $F(x) = 1 - \exp(-\lambda\, x)$ and $G(x)$ is given by (159).

The stability condition is $\lambda < \mu$, that is from (156) and (160),

$$
\sum_{k=1}^{N} \rho_k < 1. \tag{161}
$$

From now on we shall assume that (161) holds.

We are now in position to determine $\phi(\theta)$. We have

$$\begin{aligned}
\phi(\theta) &= E[e^{\theta(s_n - \tau_n)}] \\
&= E[e^{\theta s_n}]\, E[e^{-\theta \tau_n)}] \quad \text{since } s_n \text{ and } \tau_n \text{ are independent r.v.'s} \\
&= \left(\frac{\lambda}{\lambda+\theta}\right) E[e^{\theta s_n}] \quad \text{since } P(\tau_n \le x) = 1 - e^{-\lambda x}.
\end{aligned}$$

It remains to evaluate $E[e^{\theta s_n}]$. We have from (159)

$$\begin{aligned}
E[e^{\theta s_n}] &= \int_0^\infty e^{\theta y}\, dG(y) \\
&= \sum_{k=1}^{N} \frac{\lambda_k}{\lambda} \int_0^\infty e^{\theta y}\, dG_k(y).
\end{aligned}$$

Therefore,

$$\phi(\theta) = \sum_{k=1}^{N} \frac{\lambda_k}{\lambda+\theta} \int_0^\infty e^{\theta y}\, dG_k(y). \tag{162}$$

From (147) we deduce that $P(W \ge b) \le q$ if $\phi(-(\log q)/b) \le 1$, that is from (162) if

$$\sum_{k=1}^{N} \frac{\lambda_k}{\lambda - (\log q)/b} \int_0^\infty e^{-(\log q)\, y/b}\, dG_k(y) \le 1. \tag{163}$$

Let us now get back to the original **Problem 1**. Let n_i, $i = 0, 1, \ldots, K$, be integer numbers such that $\sum_{i=1}^{K} n_i = N$ and $n_0 = 0$, and assume that customers with class in $\{1 + \sum_{j=1}^{i} n_j, \ldots, \sum_{j=1}^{i+1} n_j\}$ are all identical ($i = 0, 1, \ldots, K-1$).

Using (156) it is easily seen that (163) can be rewritten as

$$\sum_{i=1}^{K} n_i\, \alpha_i \le 1 \tag{164}$$

with

$$\alpha_i := \frac{\lambda_i\, b\,(1 - \phi_i(-(\log q)/b))}{\log q} \tag{165}$$

where $\phi_i(\theta) := \int_0^\infty \exp(\theta y)\, dG_i(y)$.

Thus, a new session, that is a new class of customers, say class i, can be admitted in the system when there are already n_1 active sessions of class 1, …, n_K active sessions of class

K, if

$$\sum_{i=1}^{K} n_i\,\alpha_i + \alpha_i \le 1. \tag{166}$$

This result is called an effective bandwidth-type result since α_i may be interpreted as the *effective bandwidth* required by a session of type i. So, the decision criterion for admitting/rejecting a new session in **Problem 1** consists in adding the effective bandwidth requirements of all the active sessions in the system to the effective bandwidth of the new session and to accept this new session if and only if the sum does not exceed 1.

Let us now focus on **Problem 2**. We could follow the same approach and use the bounds in Result 7.2. We shall instead use the exact formula for $E[W]$ (see the Polloczek-Khinchin formula (60)).

We have

$$E[W] = \frac{\lambda\,\overline{\sigma}^2}{2\,(1-\rho)}$$

where $\rho := \sum_{k=1}^{N} \rho_k = \sum_{i=1}^{K} n_i\,\rho_i$ and where $\overline{\sigma}^2$ is the second-order moment of the service time of an arbitrary customer.

Let $\overline{\sigma}_i^2$ be the second-order moment of customers of class i, for $i = 1, 2, \ldots, K$. Using (159) we see that

$$\overline{\sigma}^2 = \sum_{i=1}^{K} \frac{n_i\,\lambda_i}{\lambda}\,\overline{\sigma}_i^2.$$

Hence,

$$E[W] = \frac{\sum_{i=1}^{K} n_i\,\lambda_i\,\overline{\sigma}_i^2}{2\left(1 - \sum_{i=1}^{K} n_i\,\rho_i\right)}.$$

Thus, the condition $E[W] < \alpha$ in **Problem 2** becomes

$$\sum_{i=1}^{K} n_i\,\lambda_i\,\overline{\sigma}_i^2 < 2\alpha\,\left(1 - \sum_{i=1}^{K} n_i\,\rho_i\right).$$

Rearranging terms, this is equivalent to

$$\sum_{i=1}^{K} n_i\,\alpha_i < 1 \tag{167}$$

where

$$\alpha_i = \rho_i + \lambda_i \frac{\overline{\sigma}_i^2}{2\,\alpha}$$

for $i = 1, 2, \ldots, K$.

The analytical expression for the effective bandwidth α_i is illuminating. Observe that ρ_i is the mean workload brought to the system per unit of time by a customer of type i. Therefore, the effective bandwidth requirement for session i is seen to be larger than ρ_i. Let $\alpha \to \infty$. Then, the constraint $E[W] < \alpha$ becomes $\sum_{i=1}^{K} n_i \, \rho_i < 1$, which is nothing but that the stability condition.

Effective bandwidth-type results have lately been obtained for much more general input processes than Poisson processes (that do not match well bursty traffic). Nowadays, the research activity that is going on in this area is important since there are many challenging open problems left.

8 Simulation

[8] In the next section you will find a method – known as the *Inversion Transform Method* – for generating any random variable (r.v.) with a given cumulative distribution function (c.d.f.) from a r.v. uniformly distributed in $(0, 1)$.

8.1 The Inversion Transform Method

Let U be a r.v. uniformly distributed in $(0, 1)$. Recall that

$$P(U \leq x) = P(U < x) = x \tag{168}$$

for all $x \in (0, 1)$.

Let X be a r.v. with c.f.d. $F(x) = P(X \leq x)$. Define

$$F^{-1}(y) = \inf \{x \, : \, F(x) > y\} \tag{169}$$

for $y \in (0, 1)$. Observe that $F^{-1}(y)$ is well-defined in $(0, 1)$.

Let $\epsilon > 0$. From the definition of $F^{-1}(y)$ we have that

$$F^{-1}(y) < x + \epsilon \Longrightarrow F(x + \epsilon) > y. \tag{170}$$

[8]Was taken from Chapter 7 of the book by C. H. Sauer and K. M. Chandy entitled *Computer Systems Performance Modeling*, Prentice-Hall, Englewoods-Cliff, 1981.

Indeed, we have that $F\left(F^{-1}(y)\right) > y$ by the very definition of $F^{-1}(y)$. Using now the fact that $F(x)$ is non-decreasing (since any c.d.f. is nondecreasing) we see that if $F^{-1}(y) < x + \epsilon$ then

$$F(x+\epsilon) \geq F\left(F^{-1}(y)\right) > y.$$

Letting $\epsilon \to 0$ in (170) and remembering that $F(x)$ is right-continuous, yields

$$F^{-1}(y) \leq x \Longrightarrow F(x) \geq y. \tag{171}$$

On the other hand, we see from (169) that

$$F(x) > y \Longrightarrow F^{-1}(y) \leq x. \tag{172}$$

We have the following result:

Result 8.1 (Inversion Transform Method) *Let U be a uniformly r.v. in $(0,1)$. Then, the r.v.*

$$X = F^{-1}(U) \tag{173}$$

has c.d.f. $F(x)$ for all x.

□

Proof. From (170) and (171) we have that

$$\{U < F(x)\} \subset \{F^{-1}(U) \leq x\} \subset \{U \leq F(x)\}. \tag{174}$$

Since $P(\{U < F(x)\}) = P(\{U \leq F(x)\}) = F(x)$ (cf. (168)) we deduce from (174) that $P(\{F^{-1}(U) \leq x) = F(x)$, which completes the proof (here we use the property that $P(A) \leq P(B)$ for any events A and B such that $A \subset B$).

♢

8.2 Applications: generating Bernoulli, Discrete, Exponential and Erlang r.v.'s

We shall only consider non-negative r.v.'s. Therefore, we will always assume that $x \geq 0$.

1. (Bernoulli r.v.) Let X be a Bernoulli r.v. with parameter p, namely,

$$X = \begin{cases} 0 & \text{with probability } 1-p \\ 1 & \text{with probability } p. \end{cases}$$

We have for all x

$$\begin{aligned}
F(x) &= P(X \leq x) \\
&= P(X \leq x \,|\, X = 0)P(X = 0) + P(X \leq x \,|\, X = 1)P(X = 1) \quad \text{by Bayes' formula} \\
&= 1 - p + \mathbf{1}(x \geq 1)\, p.
\end{aligned}$$

Therefore, $F^{-1}(y) = 0$ for all $0 \leq y < 1 - p$ and $F^{-1}(y) = 1$ for all $1 - p \leq y < 1$ (hint: draw the curve $x \to F(x)$) which implies from Result 8.1 that

$$X = \begin{cases} 0 & \text{if } 0 < U < 1 - p \\ 1 & \text{if } 1 - p \leq U < 1 \end{cases} \tag{175}$$

is a r.v. with c.d.f $F(x)$ for all x.

2. (Discrete r.v.) Let X be a discrete r.v. taking values in the set $\{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$, and such that $X = a_i$ with the probability p_i, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} p_i = 1$.

Using again Bayes' formula, it is easily seen that the c.d.f. of X is given by

$$F(x) = \sum_{i=1}^{n} \mathbf{1}(x \leq a_i)\, p_i.$$

Therefore, by Result 8.1

$$X = \begin{cases} a_1 & \text{if } 0 < U < p_1 \\ a_2 & \text{if } p_1 \leq U < p_1 + p_2 \\ \vdots & \\ a_n & \text{if } p_1 + p_2 + \cdots + p_{n-1} \leq U < 1 \end{cases} \tag{176}$$

is a r.v. with c.d.f. $F(x)$ for all x.

3. (Exponential r.v.) Let X be a exponential r.v. with parameter λ, namely, $F(x) = 1 - \exp(-\lambda\, x)$ for all $x \geq 0$.

Since $x \to F(x)$ is strictly increasing and continuous, in this case $F^{-1}(y)$ is simply the inverse of $F(x)$. Therefore, it is seen from Result 8.1, that the r.v. X defined by

$$X = \frac{-1}{\lambda} \log U \tag{177}$$

is a r.v. with c.d.f. $F(x)$ for all x.

4. (Erlang r.v.) A r.v. X is an Erlang $(\lambda^{-1},\ n)$ r.v. if $X = T_1 + T_2 + \ldots + T_n$, where $T_1, T_2, \ldots, T_n$ are n exponential and independent r.v.'s such that $P(T_i \leq x) = 1 - \exp(-\lambda\, x)$ for $i = 1, 2, \ldots, n$.

Let $U_1, U_2, \ldots, U_n$ be n independent r.v.'s, uniformly distributed in $(0, 1)$.

We know from 3. that $(-1/\lambda)\log U_i$ is an exponential r.v. with parameter λ. Therefore, the r.v. X defined by

$$X = \frac{-1}{\lambda}\log(U_1\, U_2 \cdots U_n) \tag{178}$$

is an Erlang (λ^{-1}, n) r.v.

Remark 8.1 (Property of an Erlang (λ^{-1}, n) r.v.) The density function of an Erlang (λ^{-1}, n) is $f_X(x) = \exp(-\lambda x)\,\lambda^k\, x^{k-1}/(k-1)!$ for $x \geq 0$. This can also be used as the definition of an Erlang (λ^{-1}, n) r.v.

The mean of an Erlang (λ^{-1}, n) r.v. is n/λ. The variance of an Erlang (λ^{-1}, n) r.v. is n/λ^2.

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Appendices

A Probability Refresher

A.1 Sample-Space, Events and Probability Measure

A *Probability space* is a triple $(\Omega, \mathcal{F}, P)$ where

- Ω is the set of all *outcomes* associated with an experiment. Ω will be called the *sample space*
- $\mathcal{F}$ is a set of subsets of Ω, called *events*, such that
  (i) $\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$
  (ii) if $A \in \mathcal{F}$ then the complementary set A^c is in $\mathcal{F}$
  (iii) if $A_n \in \mathcal{F}$ for $n = 1, 2, \ldots$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$

  $\mathcal{F}$ is called a σ-algebra.
- P is a *probability measure* on $(\Omega, \mathcal{F})$, that is, P is a mapping from $\mathcal{F}$ into $[0, 1]$ such that
  (a) $P(\emptyset) = 0$ and $P(\Omega) = 1$
  (b) $P\left(\cup_{n \in I} A_n\right) = \sum_{n \in I} P(A_n)$ for any countable (finite or infinite) family $\{A_n,\, n \in I\}$ of *mutually exclusive* events (i.e., $A_i \cap A_j = \emptyset$ for $i \in I$, $j \in I$ such that $i \neq j$).

Axioms (ii) and (iii) imply that $\cap_{n \in I} A_n \in \mathcal{F}$ for any countable (finite or infinite) family $\{A_n,\, n \in I\}$ of events (write $\cap_{n \in I} A_n$ as $(\cup_{n \in I} A_n^c)^c$). The latter result implies, in particular, that $B - A \in \mathcal{F}$ (since $B - A = B \cap A^c$).

Axioms (a) and (b) imply that for any events A and B, $P(A) = 1 - P(A^c)$ (write Ω as $A \cup A^c$), $P(A) \leq P(B)$ if $A \subset B$ (write B as $A \cup (B - A)$), $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (write $A \cup B$ as $(A \cap B) \cup (A^c \cap B) \cup (A \cap B^c)$).

Example A.1 The experiment consists in rolling a die. Then

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

$A = \{1, 3, 5\}$ is the event of rolling an odd number. Instances of σ-algebras on Ω are $\mathcal{F}_1 = \{\emptyset, \Omega\}$, $\mathcal{F}_2 = \{\emptyset, \Omega, A, A^c\}$, $\mathcal{F}_3 = \{\emptyset, \Omega, \{1, 2, 3, 5\}, \{4, 6\}\}$, $\mathcal{F}_4 = \mathcal{P}(\Omega)$ (the set of all subsets of Ω). Is $\{\emptyset, \Omega, \{1, 2, 3\}, \{3, 4, 6\}, \{5, 6\}\}$ a σ-algebra?

$\mathcal{F}_1$ and $\mathcal{F}_4$ are the smallest and the largest σ-algebras on Ω, respectively.

If the die is not biaised, the probability measure on, say, $(\Omega, \mathcal{F}_3)$, is defined by $P(\emptyset) = 0$, $P(\Omega) = 1$, $P(\{1, 2, 3, 5\}) = 4/6$ and $P(\{4, 6\}) = 2/6$.

♡

Example A.2 The experiment consists in rolling two dice. Then

$$\Omega = \{(1, 1), (1, 2), \ldots, (1, 6), (2, 1), (2, 2), \ldots, (6, 6)\}.$$

$A = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$ is the event of rolling a seven.

♡

Example A.3 The experiment consists in tossing a fair coin until a head appears. Then,

$$\Omega = \{H, TH, TTH, TTTH, \ldots\}.$$

$A = \{TTH, TTTH\}$ is the event that 3 or 4 tosses are required.

♡

Example A.4 The experiment measures the *response* time that elapses from the instant the last character of a request is entered on an inter-active terminal until the last character of the response from the computer has been received and displayed. We assume that the response time is at least of 1 second. Then,

$$\Omega = \{\text{real } t \,:\, t \geq 1\}.$$

$A = \{10 \leq t \leq 20\}$ is the event that the response time is between 10 and 20 seconds.

♡

A.2 Combinatorial Analysis

A *permutation* of order k of n elements is an ordered subset of k elements taken from the n elements.

A *combination* of order k of n elements is an unordered selection of k elements taken from the n elements.

Recall that $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ for any nonnegative integer n with $0! = 1$ by convention.

Result A.1 *The number of permutations of order k of n elements is*

$$A(n,k) = \frac{n!}{(n-k)!} = n\,(n-1)\,(n-2)\cdots(n-k+1).$$

$\square$

Result A.2 *The number of combinations of order k of n elements is*

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

$\square$

Example A.5 Suppose that 5 terminals are connected to an on-line computer system via a single communication channel, so that only one terminal at a time may use the channel to send a message to the computer. At every instant, there may be 0, 1, 2, 3, 4, or 5 terminals ready for transmission. One possible sample space is

$$\Omega = \{(x_1, x_2, x_3, x_4, x_5) \ : \ \text{each } x_i \text{ is either 0 or 1}\}.$$

$x_i = 1$ means that terminal i is ready to transmit a message, $x_i = 0$ that it is not ready. The number of points in the sample space is 2^5 since each x_i of $(x_1, x_2, x_3, x_4, x_5)$ can be selected in two ways.

Assume that there are always 3 terminals in the ready state. Then,

$$\Omega = \{(x_1, x_2, x_3, x_4, x_5) \ : \ \text{exactly 3 of the } x_i\text{'s are 1 and 2 are 0}\}.$$

In that case, the number n of points in the sample space is the number of ways that 3 ready terminals can be chosen from the 5 terminals, which is, from Result A.2,

$$\binom{5}{3} = \frac{5!}{3!\,(5-3)!} = 10.$$

Assume that each terminal is equally likely to be in the ready condition.

If the terminals are polled sequentially (i.e., terminal 1 is polled first, then terminal 2 is polled, etc.) until a ready terminal is found, the number of polls required can be 1, 2 or 3. Let A_1, A_2, and A_3 be the events that the required number of polls is 1, 2, 3, respectively.

A_1 can only occur if $x_1 = 1$, and the other two 1's occur in the remaining four positions. The number n_1 of points favorable to A_1 is calculated as $n_1 = \binom{4}{2} = 6$ and therefore $P(A_1) = n_1/n = 6/10$.

A_2 can only occur if $x_1 = 0$, $x_2 = 0$, and the remaining two 1's occur in the remaining three positions. The number n_2 of points favorable to A_1 is calculated as $n_2 = \binom{3}{2} = 3$ and therefore $P(A_1) = 3/10$.

Similarly, $P(A_3) = 1/10$.

♡

A.3 Conditional Probability

The probability that the event A occurs given the event B has occurred is denoted by $P(A \mid B)$.

Result A.3 (Bayes' formula)

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

The conditional probability is not defined if $P(B) = 0$. It is easily checked that $P(\cdot \mid B)$ is a probability measure.

□

Interchanging the role of A and B in the above formula yields

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}.$$

provided that $P(A) > 0$.

Let A_i, $i = 1, 2, \ldots, n$ be n events. Assume that the events $A_1, \ldots, A_{n-1}$ are such that $P(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) > 0$. Then,

Result A.4 (Generalized Bayes' formula)

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)\, P(A_2 \,|\, A_1) \,\cdots \times P(A_n \,|\, A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

□

The proof if by induction on n. The result is true for $n = 2$. Assume that it is true for $n = 2, 3, \ldots, k$, and let us show that it is still true for $n = k + 1$.

Define $A = A_1 \cap A_2 \cap \cdots \cap A_k$. We have

$$\begin{aligned} P(A_1 \cap A_2 \cap \cdots \cap A_{k+1}) &= P(A \cap A_{k+1}) \\ &= P(A)\, P(A_{k+1} \,|\, A) \\ &= P(A_1)\, P(A_2 \,|\, A_1) \,\cdots P(A_k \,|\, A_1 \cap A_2 \cap \cdots \cap A_{k-1}) \\ &\quad \times P(A_{k+1} \,|\, A) \end{aligned}$$

from the inductive hypothesis, which completes the proof.

■

Example A.6 A survey of 100 computer installations in a city shows that 75 of them have at least one brand X computer. If 3 of these installations are chosen at *random*, what is the probability that each of them has at least one brand X machine?

Answer: let A_1, A_2, A_3 be the event that the first, second and third selection, respectively, has a brand X computer.

The required probability is

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_1)\, P(A_2 \,|\, A_1)\, P(A_3 \,|\, A_1, A_2) \\ &= \frac{75}{100} \times \frac{74}{99} \times \frac{73}{98} \\ &= 0.418. \end{aligned}$$

♡

The following result will be used extensively throughout the course.

Result A.5 (Law of total probability) *Let $A_1, A_2, \ldots, A_n$ be events such that*

(a) $A_i \cap A_j = \emptyset$ if $i \neq j$ (mutually exclusive events)

(b) $P(A_i > 0)$ *for* $i = 1, 2, \ldots, n$

(c) $A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$.

Then, for any event A,

$$P(A) = \sum_{i=1}^{n} P(A \mid A_i)\, P(A_i).$$

□

To prove this result, let $B_i = A \cap A_i$ for $i = 1, 2, \ldots, n$. Then, $B_i \cap B_j = \emptyset$ for $i \neq j$ (since $A_i \cap A_j = \emptyset$ for $i \neq j$) and $A = B_1 \cup B_2 \cup \cdots \cup B_n$. Hence,

$$P(A) = P(B_1) + P(B_2) + \cdots + P(B_n)$$

from axiom (b) of a probability measure. But $P(B_i) = P(A \cap A_i) = P(A \mid A_i)\, P(A_i)$ for $i = 1, 2, \ldots, n$ from Bayes' formula, and therefore $P(A) = \sum_{i=1}^{n} P(A \mid A_i)\, P(A_i)$, which concludes the proof.

■

Example A.7 Requests to an on-line computer system arrive on 5 communication channels. The percentage of messages received from lines 1, 2, 3, 4, 5, are 20, 30, 10, 15, and 25, respectively. The corresponding probabilities that the length of a request will exceed 100 bits are 0.4, 0.6, 0.2, 0.8, and 0.9. What is the probability that a randomly selected request will be longer than 100 bits?

Answer: let A be the event that the selected message has more than 100 bits, and let A_i be the event that it was received on line i, $i = 1, 2, 3, 4, 5$. Then, by the law of total probability,

$$\begin{aligned} P(A) &= \sum_{i=1}^{5} P(A \mid A_i)\, P(A_i) \\ &= 0.2 \times 0.4 + 0.3 \times 0.6 + 0.1 \times 0.2 + 0.15 \times 0.8 + 0.25 \times 0.9 \\ &= 0.625. \end{aligned}$$

♡

Two events A and B are said to be *independent* if

$$P(A \cap B) = P(A)\, P(B).$$

This implies the usual meaning of independence; namely, that neither influences the occurrence of the other. Indeed, if A and B are independent, then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\, P(B)}{P(B)} = P(A)$$

and

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)\, P(B)}{P(A)} = P(B).$$

The concept of independent events should not be confused with the concept of mutually exclusive events (i.e., $A \cap B = \emptyset$). In fact, if A and B are mutually exclusive then

$$0 = P(\emptyset) = P(A \cap B)$$

and thus $P(A \cap B)$ cannot be equal to $P(A)\, P(B)$ unless at least one of the events has probability 0. Hence, mutually exclusive events *are not independent* except in the trivial case when at least one of them has zero probability.

A.4 Random Variables

In many random experiments we are interested in some number associated with the experiment rather than the actual outcome (i.e., $\omega \in \Omega$). For instance, in Example A.2 one may be interested in the sum of the numbers shown on the dice. We are thus interested in a function that associates a number with an experiment. Such function is called a *random variable* (r.v.).

More precisely, a *real-valued* r.v. X is a mapping from Ω into $\mathbb{R}$ such that

$$\{\omega \in \Omega \,:\, X(\omega) \leq x\} \in \mathcal{F}$$

for all $x \in \mathbb{R}$.

As usual, we let $X = x$ denote the event $\{\omega \in \Omega \,:\, X(\omega) = x\}$, $X \leq x$ denote the event $\{\omega \in \Omega \,:\, X(\omega) \leq x\}$, and $y \leq X \leq x$ denote the event $\{\omega \in \Omega \,:\, y \leq X(\omega) \leq x\}$.

The requirement that $X \leq x$ be an event for X to be a r.v. is necessary so that probability calculations can be made.

For each r.v. X, we define its *cumulative distribution function* (c.d.f.) F (also called the *probability distribution* of X or the *law* of X) as

$$F(x) = P(X \leq x)$$

for each $x \in \mathbb{R}$.

F satisfies the following properties: $\lim_{x\to+\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$, and $F(x) \leq F(y)$ if $x \leq y$ (i.e., F is nondecreasing).

A r.v. is *discrete* if it takes only discrete values. The distribution function F of a discrete r.v. X with values in a countable (finite or infinite) set I (e.g. $I = \mathbb{N}$) is simply given by

$$F(x) = P(X = x)$$

for each $x \in I$. We have $\sum_{x\in I} F(x) = 1$.

Example A.8 (The Bernouilli distribution) Let $p \in (0, 1)$. A r.v. variable X taking values in the set $I = \{0, 1\}$ is said to be a *Bernoulli r.v.* with parameter p, or to have a *Bernoulli distribution* with parameter p if $P(X = 1) = p$ and $P(X = 0) = 1 - p$.

♡

A r.v. is *continuous* if $P(X = x) = 0$ for all x. The *density function* of a continuous r.v. is a function f such that

(a) $f(x) \geq 0$ for all real x

(b) f is integrable and $P(a \leq X \leq b) = \int_a^b f(x)\,dx$ if $a < b$

(c) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

(d) $F(x) = \int_{-\infty}^{x} f(t)\,dt$ for all real x.

The formula $F(x) = \dfrac{\partial f(x)}{\partial x}$ that holds at each point x where f is continuous, provides a mean of computing the density function from the distribution function, and conversely.

Example A.9 (The exponential distribution) Let $\alpha > 0$. A r.v. X is said to be an *exponential r.v.* with parameter α or to have an *exponential distribution* with parameter α if

$$F(x) = \begin{cases} 1 - \exp(-\alpha x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

The density function f is given by

$$f(x) = \begin{cases} \alpha \exp(-\alpha x) & \text{if } x > 0 \\ \\ 0 & \text{if } x \leq 0. \end{cases}$$

Suppose that $\alpha = 2$ and we wish to calculate the probability that X lies in the interval $(1, 2]$. We have

$$\begin{aligned} P(1 < X \leq 2) &= P(X \leq 2) - P(X \leq 1) \\ &= F(2) - F(1) \\ &= (1 - \exp(-4)) - (1 - \exp(-2)) \\ &= 0.117019644. \end{aligned}$$

♡

Example A.10 (The exponential distribution is memoryless) Let us now derive a key feature of the exponential distribution: the fact that it is *memoryless*. Let X be an exponential r.v. with parameter α. We have

$$\begin{aligned} P(X > x + y \,|\, X > x) &= \frac{P(X > x + y,\, X > x)}{P(X > x)} \quad \text{from Bayes' formula} \\ &= \frac{P(X > x + y)}{P(X > x)} = \frac{e^{-\alpha(x+y)}}{e^{-\alpha x}} \\ &= e^{-\alpha y} \\ &= P(X > y) \end{aligned}$$

which does not depend on x!

♡

A.5 Parameters of a Random Variable

Let X be a discrete r.v. taking values in the set I.

The *mean* or the *expectation* of X, denoted as $E[X]$, is the number

$$E[X] = \sum_{x \in I} x \, P(X = x)$$

provided that $\sum_{x \in I} |x| \, P(X = x) < \infty$.

Example A.11 (Expectation of a Bernoulli r.v.) Let X be a Bernoulli r.v. with parameter p. Then,

$$
\begin{aligned}
E[X] &= 0 \times P(X = 0) + 1 \times P(X = 1) \\
&= p.
\end{aligned}
$$

♡

If X is a continuous r.v. with density function function f, we define the expectation or the mean of X as the number

$$E[X] = \int_{-\infty}^{+\infty} x\, f(x)\, dx$$

provided that $\int_{-\infty}^{+\infty} |x|\, f(x)\, dx < \infty$.

Example A.12 (Expectation of an exponential r.v.) Let X be an exponential r.v. with parameter $\alpha > 0$. Then,

$$
\begin{aligned}
E[X] &= \int_{0}^{+\infty} x\, \alpha\, \exp(-\alpha x)\, dx \\
&= \frac{1}{\alpha}.
\end{aligned}
$$

by using an integration by parts (use the formula $\int u dv = uv - \int v du$ with $u = x$ and $dv = \alpha \exp(-\alpha x) dx$, together with $\lim_{x \to +\infty} x \exp(-\alpha x) = 0$).

♡

Let us give some properties of the expectation operator $E[\cdot]$.

Result A.6 *Suppose that X and Y are r.v.'s. such that $E[X]$ and $E[Y]$ exist, and let c be a real number. Then, $E[c] = c$, $E[X + Y] = E[X] + E[Y]$, and $E[cX] = c\, E[X]$.*

□

The *k-th moment* or the *moment of order k* ($k \geq 1$) of a discrete r.v. X taking values in the set I is given by

$$E[X^k] = \sum_{x \in I} x^k\, P(X = x)$$

provided that $\sum_{x \in I} |x^k|\, P(X = x) < \infty$.

The k-th moment or the moment of order k ($k \geq 1$) of a continuous r.v. X is given by

$$E[X^k] = \int_{-\infty}^{+\infty} x^k\, f(x)\, dx$$

provided that $\int_{-\infty}^{+\infty} |x^k|\, f(x)\, dx < \infty$.

The variance of a discrete or continuous r.v. X is defined to be

$$\mathrm{var}\,(X) = E\,(X - E[X])^2 = E[X^2] - (E[X])^2.$$

Example A.13 (Variance of the exponential distribution) Let X be an exponential r.v. with parameter $\alpha > 0$. Then,

$$\begin{aligned} \mathrm{var}\,(X) &= \int_0^{+\infty} x^2\, \alpha\, \exp(-\alpha x)\, dx - \frac{1}{\alpha^2} \\ &= \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}. \end{aligned}$$

Hence, the variance of an exponential r.v. is the square of its mean.

♡

Example A.14 Consider the situation described in Example A.5 when the terminals are polled sequentially until one terminal is found ready to transmit. We assume that each terminal is ready to transmit with the probability p, $0 < p \leq 1$, when it is polled.

Let X be the number of polls required before finding a terminal ready to transmit. Since $P(X = 1) = p$, $P(X = 2) = (1 - p)\, p$, and more generally since $P(X = n) = (1 - p)^{n-1}\, p$ for each n, we have

$$E[X] = \sum_{n=1}^{\infty} n\,(1 - p)^{n-1}\, p = 1/p.$$

Observe that $E[X] = 1$ if $p = 1$ and $E[X] \to \infty$ when $p \to 0$ which agrees with intuition.

♡

A.6 Jointly Distributed Random Variables

Sometimes it is of interest to investigate two or more r.v.'s. If X and Y are defined on the same probability space, we define the *joint cumulative distribution function* (j.c.d.f.) of X and Y for all real x and y by

$$F(x, y) = P(X \leq x,\, Y \leq y) = P((X \leq x) \cap (Y \leq y)).$$

Define $F_X(x) = P(X \le x)$ and $F_Y(y) = P(Y \le y)$ for all real x and y. F_X and F_Y are called the *marginal cumulative distribution functions* of X and Y, respectively, corresponding to the joint distribution function F.

Note that $F_X(x) = \lim_{y \to +\infty} F(x, \infty)$ and $F_Y(y) = \lim_{x \to +\infty} F(\infty, y)$.

If there exists a nonnegative function f of two variables such that

$$P(X \le x,\, Y \le Y] = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, du\, dv$$

then f is called the *joint density function* of the r.v.'s X and Y.

Suppose that g is a function of two variables and let f be the joint density function of X and Y. The expectation $E[g(X, Y)]$ is defined as

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\, f(x, y)\, dx\, dy$$

provided that the integral exists.

Consider now the case when X and Y are discrete r.v.'s taking values in some countable sets I and J, respectively. Then the *joint distribution function* of X and Y for all $x \in I$, $y \in J$, is given by

$$F(x, y) = P(X = x,\, Y = y) = P((X = x) \cap (Y = y)).$$

Define $F_X(x) = P(X = x)$ and $F_Y(y) = P(Y = y)$ for all $x \in I$ and $y \in J$ to be the marginal distribution functions of X and Y, respectively, corresponding to the joint distribution function F.

From the law of total probability, we see that $F_X(x) := \sum_{y \in J} F(x, y) = P(X = x)$ for all $x \in I$ and $F_Y(y) := \sum_{x \in I} F(x, y) = P(Y = y)$ for all $y \in J$.

Suppose that g is a nonnegative function of two variables. The expectation $E[g(X, Y)]$ is defined as

$$E[g(X, Y)] = \sum_{x \in I,\, y \in J} g(x, y)\, P(X = x,\, Y = y)$$

provided that the summation exists.

A.7 Independent Random Variables

Two r.v.'s X and Y are said to be *independent* if

$$P(X \le x,\, Y \le y) = P(X \le x)\, P(Y \le y)$$

for all real x and y if X and Y are continuous r.v.'s, and if

$$P(X = x,\, Y = y) = P(X = x)\, P(Y = y)$$

for all $x \in I$ and $y \in J$ if X and Y are discrete and take their values in I and J, respectively.

Result A.7 *If X and Y are independent r.v.'s such that $E[X]$ and $E[Y]$ exist, then*

$$E[X\,Y] = E[X]\,E[Y].$$

□

Let us prove this result when X and Y are discrete r.v.'s taking values in the sets I and J, respectively. Let $g(x, y) = xy$ in the definition of $E[g(X, Y)]$ given in the previous section. Then,

$$\begin{aligned}
E[X\,Y] &= \sum_{x \in I,\, y \in J} xyP(X = x,\, Y = y) \\
&= \sum_{x \in I,\, y \in J} xy\, P(X = x)\, P(Y = y) \quad \text{since } X \text{ and } Y \text{ are independent r.v.'s} \\
&= \sum_{x \in I} x\, P(X = x) \left(\sum_{y \in I} y\, P(Y = y) \right) \\
&= E[X]\,E[Y].
\end{aligned}$$

The proof when X and Y are both continuous r.v.'s is anologous and therefore omitted.

□

A.8 Conditional Expectation

Consider the situation in Example A.5. Let X be the number of polls required to find a ready terminal and let Y be the number of ready terminals. The mean number of polls given that $Y = 1, 2, 3, 4, 5$ is the conditional expectation of X given Y (see the computation in Example A.15).

Let X and Y be discrete r.v.'s with values in the sets I and J, respectively.

Let $P_{X|Y}(x, y) := P(X = x \,|\, Y = y)$ be the conditional probability of the event $(X = x)$ given the event $(Y = y)$. From Result A.3 we have

$$P_{X|Y}(x, y) = \frac{P(X = x,\, Y = y)}{P(Y = y)}$$

for each $x \in I$, $y \in J$, provided that $P(Y = y) > 0$.

$P_{X|Y}(\cdot \,|\, y)$ is called the *conditional distribution function* of X given $Y = y$.

The *conditional expectation* of X given $Y = y$, denoted by $E[X \,|\, Y = y]$, is defined for all $y \in J$ such that $P(Y = y) > 0$ by

$$E[X \,|\, Y = y] = \sum_{x \in I} x \, P_{X|Y}(x, y)$$

Example A.15 Consider Example A.5. Let $X \in \{1, 2, 3, 4, 5\}$ be the number of polls required to find a terminal in the ready state and let $Y \in \{1, 2, 3, 4, 5\}$ be the number of ready terminals. We want to compute $E[X \,|\, Y = 3]$, the mean number of polls required given that $Y = 3$.

We have

$$\begin{aligned} E[X \,|\, Y = 3] &= 1 \times P_{X|Y}(1, 3) + 2 \times P_{X|Y}(2, 3) + 3 \times P_{X|Y}(3, 3) \\ &= 1 \times P(A_1) + 2 \times P(A_2) + 3 \times P(A_3) \\ &= \frac{6}{10} + 2 \times \frac{3}{10} + 3 \times \frac{1}{10} = \frac{15}{10} = 1.5\,. \end{aligned}$$

The result should not be surprising since each terminal is equally likely to be in the ready state.

♡

Consider now the case when X and Y are both continuous r.v.'s with density functions f_X and f_Y, respectively, and with joint density function f. The *conditional probability density function* of X given $Y = y$, denoted as $f_{X|Y}(x, y)$, is defined for all real y such that $f_Y(y) > 0$, by

$$f_{X|Y}(x, y) = \frac{f(x, y)}{f_Y(y)}.$$

The conditional expectation of X given $Y = y$ is defined for all real y such that $f_Y(y) > 0$, by

$$E[X \,|\, Y = y] = \int_{-\infty}^{+\infty} x \, f_{X|Y}(x, y) \, dx.$$

Below is a very useful result on conditional expectation. This is the version of the law of total probability for the expectation.

Result A.8 (Law of conditional expectation) *For any r.v.'s X and Y,*

$$E[X] = \sum_{y \in J} E[X \,|\, Y = y]\, P(Y = y)$$

if X is a discrete r.v., and

$$E[X] = \int_{-\infty}^{\infty} E[X \,|\, Y = y]\, f_Y(y)\, dy$$

if X is a continuous r.v.

□

We prove the result in the case when X and Y are both discrete r.v.'s. Since $E[X \,|\, Y = y] = \sum_{x \in I} x\, P(X = x \,|\, Y = y)$ we have from the definition of the expectation that

$$\begin{aligned}
\sum_{y \in J} E[X \,|\, Y = y]\, P(Y = y) &= \sum_{y \in J} \left(\sum_{x \in I} x\, P(X = x \,|\, Y = y) \right) P(Y = y) \\
&= \sum_{x \in I} x \left(\sum_{y \in J} P(X = x \,|\, Y = y)\, P(Y = y) \right) \\
&= \sum_{x \in I} x\, P(X = x) \quad \text{by using the law of total probability} \\
&= E[X].
\end{aligned}$$

The proof in the case when X and Y are continuous r.v.'s is analogous and is therefore omitted. ■

B Stochastic Processes

B.1 Definitions

All r.v.'s considered from now on are assumed to be constructed on a common probability space $(\Omega, \mathcal{F}, P)$.

Notation: We denote by $\mathbb{N}$ the set of all nonnegative integers and by $\mathbb{R}$ the set of all real numbers.

A collection of r.v.'s $\mathbf{X} = (X(t),\, t \in T)$ is called a *stochastic process*. In other words, for each $t \in T$, $X(t)$ is a mapping from Ω into some set E where $E = \mathbb{R}$ or $E \subset \mathbb{R}$ (e.g.,

$E = [0, \infty)$, $E = \mathbb{N}$) with the interpretation that $X(t)(\omega)$ (also written $X(t, \omega)$) is the value of the stochastic process $\mathbf{X}$ at time t on the outcome (or path) ω.

The set T is the *index set* of the stochastic process. If T is countable (e.g., if $T = \mathbb{N}$ or $T = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$), then $\mathbf{X}$ is called a *discrete-time* stochastic process; if T is continuous (e.g., $T = \mathbb{R}$, $T = [0, \infty)$) then $\mathbf{X}$ is called a *continuous-time* stochastic process. When T is countable we will usually substitute the notation X_n (or $X(n)$, t_n, etc.) for $X(t)$.

The space E is called the *state space* of the stochastic process $\mathbf{X}$. If the set E is countable then $\mathbf{X}$ is called a *discrete-space* stochastic process; if the set E is continuous then $\mathbf{X}$ is called a *continuous-space* stochastic process.

When speaking of "the process $X(t)$" one should understand the process $\mathbf{X}$. This is a common abuse of language.

Example B.1 (Discrete-time, discrete-space stochastic process) $X_n =$ number of jobs processed during the n-th hour of the day in some job shop. The stochastic process $(X_n,\ n = 1, 2, \ldots, 24)$ is a discrete-time, discrete-space stochastic process.

♡

Example B.2 (Discrete-time, continuous-space stochastic process) $X_n =$ response time of the n-th inquiry to the central processing system of an interactive computer system. The stochastic process $(X_n,\ n = 1, 2, \ldots)$ is a discrete-time, continuous-space stochastic process.

♡

Example B.3 (Continuous-time, discrete-space stochastic process) $X(t) =$ number of messages that have arrived at a given node of a communication network in the time period $(0, t)$. The stochastic process $(X(t),\ t \geq 0)$ is a continuous-time, discrete-space stochastic process.

♡

Example B.4 (Continuous-time, continuous-space stochastic process) $X(t) =$ waiting time of an inquiry received at time t. The stochastic process $(X(t),\ t \geq 0)$ is a continuous-time, continuous-space stochastic process.

♡

Introduce the following notation: a function f is $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0.$$

For instance, $f(h) = h^2$ is $o(h)$, $f(h) = h$ is not, $f(h) = h^r$, $r > 1$, is $o(h)$, $f(h) = \sin(h)$ is not. Any linear combination of $o(h)$ functions is also $o(h)$.

Example B.5 Let X be an exponential r.v. with parameter λ. In other words, $P(X \leq x) = 1 - \exp(-\lambda x)$ for $x \geq 0$ and $P(X \leq x) = 0$ for $x < 0$. Then, $P(X \leq h) = \lambda\, h + o(h)$.

Similarly, $P(X \leq t + h \,|\, X > t) = \lambda\, h + o(h)$ since $P(X \leq t + h \,|\, X > t) = P(X \leq h)$ from the memoryless property of the exponential distribution.

♡

C Poisson Process

A *Poisson process* is one of the simplest interesting stochastic processes. It can be defined in a number of equivalent ways. The following definition seems most appropriate to our needs.

Consider the discrete-time, continuous-space stochastic process $(t_n,\ n = 1, 2, \ldots)$ where $0 \leq t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots$. The r.v. t_n records the occurrence time of the n-th event in some experiment. For instance, t_n will be the arrival time of the n-th request to a database.

This stochastic process is a *Poisson process* with *rate* $\lambda > 0$ if:

(a) Prob (a single event in an interval of duration h) $= \lambda h + o(h)$

(b) Prob (more than one event in an interval of duration h) $= o(h)$

(c) The numbers of events occurring in nonoverlapping intervals of time are independent of each other.

From this definition and the axioms of probability, we see that

(d) Prob (no events occur in an interval of length h) $= 1 - \lambda h + o(h)$.

With a slight abuse of notation, $N(t)$, the number of events of a Poisson process in an interval of length t, is sometimes called a Poisson process. Processes like the (continuous-time, discrete-space) stochastic process $(N(t),\, t \geq 0)$ (or simply $N(t)$) are called *counting processes*.

One of the original applications of the Poisson process in communications was to model the arrivals of calls to a telephone exchange (the work of A. K. Erlang in 1919). The aggregate use of telephones, at least in a first analysis, can be modeled as a Poisson process.

Here is an important result:

Result C.1 *Let $P_n(t)$ be the probability that exactly n events occur in an interval of length t, namely, $P_n(t) = P(N(t) = n)$. We have, for each $n \in \mathbb{N}$, $t \geq 0$,*

$$P_n(t) = \frac{(\lambda t)^n}{n!}\, e^{-\lambda t}. \tag{179}$$

□

We prove this result by using an induction argument. Because the number of events in the interval $[0, a)$ is independent of the number of events in the interval $[a, a+t]$, we may simply consider the interval of time $[0, t]$.

We have

$$\begin{aligned} P_0(t+h) &= P(\text{no events in } [0,t),\ \text{no events in } [t, t+h]) \\ &= P_0(t)\, P_0(h) \quad \text{from property (c) of a Poisson process} \\ &= P_0(t)\,(1 - \lambda h + o(h)) \quad \text{from property (d) of a Poisson process.} \end{aligned}$$

Therefore, for $h > 0$,

$$\frac{P_0(t+h) - P_0(t)}{h} = -\lambda\, P_0(t) + \frac{o(h)}{h}.$$

Letting $h \to 0$ in both sides yields the ordinary differential equation

$$\frac{dP_0(t)}{dt} = -\lambda\, P_0(t).$$

The solution is $P_0(t) = \exp(-\lambda t)$ by noting that the initial condition is $P_0(0) = 1$.

Let $n \geq 1$ be an arbitrary integer and assume that (179) is true for the first $(n-1)$ integers. We have

$$P_n(t+h) = \sum_{k=0}^{n} P(\text{exactly } k \text{ events in } [0,t),\ \text{exactly } n-k \text{ events in } [t, t+h]) \tag{180}$$

$$= \sum_{k=0}^{n} P_k(t)\, P_{n-k}(h) \tag{181}$$

$$= P_n(t)\, P_0(h) + P_{n-1}(t)\, P_1(h) + o(h) \tag{182}$$

$$= P_n(t)\,(1 - \lambda h) + P_{n-1}(t)\lambda h + o(h) \tag{183}$$

where (180) follows from the law of total probability, and where (181)-(183) follow from properties (a)-(d) of a Poisson process.

Therefore, for $h > 0$,

$$\frac{P_n(t+h) - P_n(t)}{h} = -\lambda\, P_n(t) + \lambda\, P_{n-1}(t) + o(h).$$

Taking the limit $h \to 0$ yields

$$\frac{dP_n(t)}{dt} = -\lambda\, P_n(t) + \lambda\, P_{n-1}(t). \tag{184}$$

Substituting (179) for $P_{n-1}(t)$ in (184), and solving the differential equation yields the desired result (one may also check the result by direct substitution).

■

Let us now compute the mean number of events of a Poisson process with rate λ in an interval of length t.

Result C.2 (Mean number of events in an interval of length t) *For each $t \geq 0$*

$$E[N(t)] = \lambda\, t.$$

□

Result C.2 says, in particular, that the mean number of events per unit of time, or equivalently, the rate at which the events occur, is given by $E[N(t)]/t = \lambda$. This is why λ is called the rate of the Poisson process.

Let us prove Result C.2. We have by using Result C.1

$$\begin{aligned}
E[N(t)] &= \sum_{k=0}^{\infty} k\, P_k(t) = \sum_{k=1}^{\infty} k\, P_k(t) \\
&= \left(\sum_{k=1}^{\infty} k\, \frac{(\lambda t)^k}{k!} \right) e^{-\lambda t} \\
&= \lambda t \left(\sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} \right) e^{-\lambda t} \\
&= \lambda t.
\end{aligned}$$

■

There is actually a strong connection between a Poisson process and the exponential distribution. Why this?

Consider the time τ that elapses between the occurrence of two consecutive events in a Poisson process.

We have the following remarkable result:

Result C.3 *For each* $x \geq 0$

$$P(\tau \leq x) = 1 - e^{-\lambda x}.$$

□

The proof is very simple. We have

$$\begin{aligned} P(\tau > x) &= P(\text{no event in an interval of length } x) \\ &= e^{-\lambda x} \end{aligned}$$

from Result C.1. Therefore, $P(\tau \leq x) = 1 - \exp(-\lambda x)$ for $x \geq 0$.

■

So, the interevent distribution of a Poisson process with rate λ is an exponential distribution with parameter λ. The Poisson process and the exponential distribution often lead to tractable models and therefore have a special place in queueing theory and performance evaluation.

We also report the following result:

Result C.4 *Let* τ_n *be the time between the* n*-th and the* $(n+1)$*-st event of a Poisson process with rate* λ*.*

The r.v.'s τ_m *and* τ_n *are independent for each* m*,* n *such that* $m \neq n$*.*

□

The proof is omitted.

In summary, the sequence $(\tau_n, n = 1, 2, \ldots)$ of interevent times of a Poisson process with rate λ is a sequence of *mutually independent* r.v.'s, each being exponentially distributed with parameter λ.

Result C.5 *The superposition of two independent Poisson processes with rates* λ_1 *and* λ_2 *is a Poisson process with rate* $\lambda_1 + \lambda_2$.

$\square$

The proof is omitted.

Example C.1 Consider the failures of a link in a communication network. Failures occur according to a Poisson process with rate 2.4 per day. We have:

(i) $P(\text{time between failures } \leq T \text{ days }) = 1 - e^{-2.4T}$

(ii) $P(k \text{ failures in } T \text{ days }) = \dfrac{(2.4T)^k}{k!}\, e^{-2.4T}$

(iii) Expected time between two consecutive failures = 10 hours

(iv) $P(0 \text{ failures in next day}) = e^{-2.4}$

(v) Suppose 10 hours have elapsed since the last failure. Then,

Expected time to next failure = 10 hours (memoryless property).

$\heartsuit$