First Midterm, SIEO3658, Fall 2005

NOTE: ONLY THE COURSE NOTES SENT TO YOU BY EMAIL CAN BE CONSULTED DURING THE EXAM.

1. (13 points) A building has 60 occupants consisting of 15 women and 45 men. The men have probability 1/2 of being colorblind and the women have probability 1/3 of being colorblind.

   i) Suppose you choose uniformly at random a person from the 60 in the building. What is the probability that the person will be colorblind?
   Ans:

$$\begin{aligned} W &= \{\text{the one selected is a woman}\} \\ M &= \{\text{the one selected is a man}\} \\ B &= \{\text{the one selected is colorblind}\} \\ P(W) &= 15/60 = 1/4 \\ P(M) &= 3/4 \\ P(B \mid W) &= 1/3 \\ P(B \mid M) &= 1/2 \end{aligned}$$

   Therefore

$$P(B) = P(B \mid W)P(W) + P(B \mid M)P(M) = 11/24.$$

   ii) Use Bayes' rule to determine the conditional probability that you chose a woman given that the occupant you chose is colorblind.
   Ans:

$$P(W \mid B) = P(B \mid W)P(W)/P(B) = 1/3 \cdot 1/4/(11/24) = 2/11.$$

2. (13 points) Consider an experiment in which a fair coin is tossed until it first yields the same result twice in succession – (heads,heads) or (tails,tails) – at which point the experiment ends.

   i) Describe the sample space for this experiment. (Note that it is infinite since there is no bound on the number of tosses before getting 2 heads in a row or 2 tails in a row.)

   ii) What is the probability that the experiment ends after an even number of tosses?

   Ans: i) The sample space is all the sequences alternating between heads and tails except for the last two tosses which are the same.

$$\{HH, TT, HTT, THH, HTHH, THTT, HTHTT, THTHH, \cdots\}.$$

ii) Let N be the number of tosses in the experiment.

$$P(N=2) = 2\cdot(1/2\cdot 1/2) = 1/2 \quad \text{corresponding to HH and TT}$$
$$P(N=4) = 2\cdot(1/2)^4 = 1/8 \quad \text{corresponding to HTHH and THTT}$$
$$P(N=2k) = 2\cdot(1/4)^k \quad \text{following the above argument}$$

Therefore

$$P(\text{the game ends after an even number of tosses}) = \sum_{k=1}^{\infty} 2\cdot(1/4)^k = (1/2)/(1-1/4) = 2/3.$$

3. (13 points) Bob and Alice repeatedly play a game of chance which Alice has probability p of winning. Each time she loses she gives a dollar to Bob, and each time she wins she receives a dollar from Bob. There are never ties.

   i) Show that the pmf of Alice's fortune W after 10 games (the number of dollars received from Bob minus the number of dollars she has given to Bob) is given by

$$\begin{aligned} P(W=2u) &= \binom{10}{5+u} p^{5+u}(1-p)^{5-u}, \quad -5 \le u \le 5 \\ &= 0, \text{ otherwise.} \end{aligned}$$

   and verify that it satisfies the normalization axiom.

   ii) Find the expected value of W and show that the variance is maximized when $p = 1/2$.

   Ans: i) Let N be the number of games she wins, so she loses $10-N$ times, which means her fortune will be $W = N-(10-N) = 2N-10$ at the end; in other words

$$P(W=2u) = P(2N-10=2u) = P(N=5+u).$$

   Now define $U = N-5$ where $-5 \le u \le 5$, we have

$$P(N=5+u) = P(U=u),$$

   It is easy to see that

$$P(N=k) = \binom{10}{k} p^k (1-p)^{10-k}, \ 0 \le k \le 10,$$

   which is equivalent to:

$$P(U=u) = P(N=5+u) = \binom{10}{5+u} p^{5+u}(1-p)^{5-u} = P(W=2u), \ -5 \le u \le 5.$$

ii) Since

$$P(N = k) = \binom{10}{k} p^k (1-p)^{10-k},\ 0 \le k \le 10,$$

we have

$$E(N) = 10p, Var(N) = 10 \cdot p(1-p).$$

Now $W = 2N - 10$ so

$$E(W) = 2 \cdot E(N) - 10 = 20p - 10,$$

and

$$Var(W) = 4 \cdot Var(N) = 40p(1-p),$$

since $\sqrt{a \cdot b} \le \frac{a+b}{2}$ and the equality only holds when $a = b$, we obtain

$$p(1-p) \le (\frac{p+1-p}{2})^2$$

and the equality only holds when $p = 1/2$; hence the variance is maximized when $p = 1/2$.

4. (11 points) A random message is received as a sequence of n bits, each of which is independently and equally likely to be 0 or 1. Show that the event that a message with no two consecutive 1's is received has probability $g_n/2^n$, where g_n satisfies

$$g_n = g_{n-1} + g_{n-2},\ n \geq 2,$$

with the initial conditions $g_0 = 1, g_1 = 2$. We note in passing that the sequence $0, 1, g_0, g_1, \ldots$ is the Fibonacci sequence. (Hint: Consider the first bit (or two) and then argue about the number of similar sequences in the remaining bits.)

Ans: Let Ω_n be all the messages of length n which contain no two consecutive 1's.

$$| \Omega_n |= g_n,$$

$$\Omega_0 = \phi \text{ and } \Omega_1 = \{0, 1\}.$$

We need to establish a recursive equation. Consider a sequence $\{X_1, X_2, \cdots, X_n\} \in \Omega_n$ where $n \geq 2$,

(a) If $X_1 = 0$, then the remaining bits $\{X_2, X_3, \cdots, X_n\}$ can be any message in Ω_{n-1}.

(b) If $X_1 = 1$, then X_2 can only be 0, and $\{X_3, X_4, \cdots, X_n\}$ can be any message in Ω_{n-2}.

Therefore we have

$$g_n = g_{n-1} + g_{n-2},\ n \geq 2.$$

Since n bits can form 2^n messages, the probability that a message has no two consecutive 1's is $g_n/2^n$.