The Scale Representation

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Abstract—**We consider "scale" a physical attribute of a signal and develop its properties. We present an operator which represents scale and study its characteristics and representation. This allows one to define the scale transform and the energy scale density spectrum which is an indication of the intensity of scale values in a signal. We obtain explicit expressions for the mean scale, scale bandwidth, instantaneous scale, and scale group delay. Furthermore, we derive expressions for mean time, mean frequency, duration, frequency bandwidth in terms of the scale variable. The short-time transform is defined and used to obtain the conditional value of scale for a given time. We show that as the windows narrows one obtains instantaneous scale. Convolution and correlation theorems for scale are derived. A formulation is devised for studying linear scale-invariant systems. We derive joint representations of time-scale and frequency-scale. General classes for each are presented using the same methodology as for the time-frequency case. As special cases the joint distributions of Marinovich–Altes and Bertrand–Bertrand are recovered. Also, joint representations of the three quantities, time-frequency-scale are devised. A general expression for the local scale autocorrelation function is given. Uncertainty principles for scale and time and scale and frequency are derived.**

I. Introduction

SCALE is a physical attribute of a signal just like frequency. For a given signal we may properly ask what is its frequency content. Similarly, we may properly ask what is its scale content. In the frequency case, we determine the frequency content via the Fourier transform; for scale we need a transform which indicates the moment of scale in the signal.

It is the purpose of this paper to develop the concept of scale, its representation and properties. The important first step is to have the scale operator and obtain the scale transform. The solution of the eigenvalue problem for this operator gives the scale transform. Subsequently we study this transform and its properties. In analogy with instantaneous frequency and group delay, we will introduce similar concepts for scale. We obtain explicit expressions for mean scale and scale bandwidth and also for instantaneous scale and scale group delay. Convolution and correlation theorems will be derived. We develop methods for obtaining joint representations involving scale [1]–[6] which are generalizations of the method devised by the author for the time-frequency case [8]–[10]. Also, we derive a joint representation involving time-frequency-scale. A fundamental attribute of frequency and time is their relation with the translation of functions and this leads to the study of linear shift invariant systems. The fundamental property of scale is that of compression; hence, we develop the methodology for linear scale invariant systems. We show that the scale operator does not commute with the time operator or frequency operator. This implies that there must be an uncertainty principle between time-scale and frequency-scale. We will derive these uncertainty principles and obtain the minimum uncertainty signal.

The use of operator methods in signal analysis has recently been devised for arbitrary physical variables [1], [2], [7]. The basic method is based on the work of Scully and Cohen [11]. We apply this method to be the study of scale.

Scaling has been an imoprtant concept in a number of fields. In the general theory of wavelets the concept of scale enters in a fundamental way. This has led to many issues regarding the concept "scale" and in particular has led to ideas regarding joint representations of time and scale. Scaled functions also appear in radar and sonar and in particular in the study of the wide band ambiguity function and also in the study of optical imaging. See [12]–[19] and references therein.

II. Representation of Signals and Operator Methods

The basic reason for expressing a signal in another domain or representation is that very often that will reveal properties of the signal which may not be apparent in the time domain. Another important reason is that the production, propagation, or behavior of a signal may depend upon physical quantities other than time. Therefore to see how a signal behaves under those circumstances one expresses the signal in the representation in question, does the analysis, and then transforms back to the time domain. In addition, if we want to construct signals with characteristics relating to a certain physical property, we obviously construct the signal in the representation of that quantity and then transform back to the time domain. Representations are usually associated with a physical quantity. In the case of the Fourier representation, the physical quantity is frequency. In the same manner we will develop the scale representation which will represent a signal in a domain for which the fundamental physical

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variable is scale. We have recently developed a general approach for associating physical quantities with operators [1], [6], [7] similar to the methods used in other fields [20], [21]. This leads to a methodology for constructing and studying representations, particularly joint representations. We review here some basic operator methods.

A. Operators

Suppose we have an operator $\mathcal{A}$ which represents or is associated with a physical quantity.[1] Operators which represent a physical quantity will generally be Hermitian. That means that for any two functions $f(t)$ and $g(t)$,

$$\int g^*(t)\mathcal{A}f(t)\,dt = \int f(t)(\mathcal{A}g(t))^*\,dt. \tag{2.1}$$

Now, if the operator is Hermitian then the solution of the eigenvalue problem

$$\mathcal{A}u(a, t) = au(a, t) \tag{2.2}$$

results in the eigenfunctions, $u(a, t)$ and eigenvalues, a. Since the operator is Hermitian the eigenvalues are real and the eigenfunctions are complete. This implies that

$$\int u(a, t)u^*(a, t')\,da = \delta(t - t');$$
$$\int u(a, t)u^*(a', t)\,dt = \delta(a - a'). \tag{2.3}$$

Therefore, any time function can be expressed as

$$f(t) = \int F(a)u(a, t)\,da \tag{2.4}$$

where $u(a, t)$ is the transformation matrix or basis kernel and $F(a)$ is the representation of the time function in the a domain. The inverse transformation is

$$F(a) = \int f(t)u^*(a, t)\,dt. \tag{2.5}$$

We shall use the double arrow to signify correspondence of the time function and its transform, $f(t) \Leftrightarrow F(a)$.

The time and frequency operators $\mathcal{T}$ and $\mathcal{W}$ are [10]

$$\mathcal{T} = t \quad \mathcal{W} = -j\frac{d}{dt} \quad \text{(time domain)}, \tag{2.6}$$

$$\mathcal{T} = j\frac{d}{d\omega}; \quad \mathcal{W} = \omega \quad \text{(frequency domain)}. \tag{2.7}$$

They satisfy the fundamental commutation relation [10]

$$[\mathcal{T}, \mathcal{W}] = \mathcal{T}\mathcal{W} - \mathcal{W}\mathcal{T} = j. \tag{2.8}$$

Operators representing other physical variables will usually be expressed in terms of the time and frequency operators. As we will see that is the case for the scale operator.

B. Averages

Perhaps the most important reason for the power of the operator method is that one can calculate averages in a very easy way. If we have a function of a, $g(a)$, then its average is[2]

$$\langle g\rangle = \int g(a)\,|F(a)|^2\,da. \tag{2.9}$$

To evaluate, one first obtains the transform $F(a)$ and then evaluates the integral. However, one does not have to find $F(a)$!—it can be done *directly* from the time function by way of

$$\langle g\rangle = \int f^*(t)g(\mathcal{A})f(t)\,dt. \tag{2.10}$$

The equality of the right hand sides of (2.9) and (2.10) is well known; we give a simple proof here for those readers not familiar with this result. Substituting, $f(t)$ as given by (2.4) into (2.10) we have

$$\langle g\rangle = \iiint F^*(a')u^*(a', t)g(\mathcal{A})F(a)u(a, t)\,dt\,da'\,da. \tag{2.11}$$

Now, $g(\mathcal{A})u(a, t) = g(a)u(a, t)$ and hence,

$$\langle g\rangle = \iiint F^*(a')u^*(a', t)g(a)F(a)u(a, t)\,dt\,da'\,da \tag{2.12}$$

$$= \iint F^*(a')g(a)F(a)\,\delta(a - a')\,da'\,da$$
$$= \int g(a)\,|F(a)|^2\,da. \tag{2.13}$$

C. Characteristic Function Operator

For a density $P(a)$, the characteristic function of a is defined by

$$M(\alpha) = \langle e^{j\alpha a}\rangle = \int e^{j\alpha a}P(a)\,da. \tag{2.14}$$

The characteristic function uniquely determines the distribution,

$$P(a) = \frac{1}{2\pi}\int M(\alpha)e^{-j\alpha a}\,d\alpha. \tag{2.15}$$

Since the characteristic function is an average, the average of $e^{j\alpha a}$, it can be calculated using (2.10),

$$M(\alpha) = \int f^*(t)e^{j\alpha\mathcal{A}}f(t)\,dt, \tag{2.16}$$

[1] Operators will be denoted by calligraphic letters. The commutator of two operators, $\mathcal{A}$ and $\mathcal{B}$, will be denoted by the standard notation, $[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$. Indefinite integrals imply an appropriate region of integration which depends on the representation being considered. Since the scale operator will give a continuous spectrum we explain the basic method assuming a continuous spectrum. In (2.2) we have assumed that the operator is expressed in the time representation, although the eigenvalue problem can be solved in any representation. In that case the operator must be expressed in the representation of interest. Also, we assume basis sets are self reciprocal.

[2] The reason for taking $|F(a)|^2$ to be the density is discussed in detail in references [1], [2], [7].

where $\mathcal{A}$ is the operator corresponding to the variable a. We call $e^{j\alpha\mathcal{A}}$ the characteristic function operator. To show explicitly that ⊙ leads to the density given by $|F(a)|^2$ express $f(t)$ in terms of its transform to obtain

$$M(\alpha) = \iiint F^*(a')u^*(a', t)e^{j\alpha\mathcal{A}} \cdot F(a)u(a, t)\, da'\, da\, dt. \tag{2.17}$$

But $e^{j\alpha\mathcal{A}}u(a, t) = e^{j\alpha a}u(a, t)$, giving

$$\begin{aligned} M(\alpha) &= \iint F^*(a')e^{j\alpha a}\,\delta(a - a')F(a)\, da'\, da \\ &= \int |F(a)|^2 e^{j\alpha a}\, da, \end{aligned} \tag{2.18}$$

which shows that

$$P(a) = |F(a)|^2. \tag{2.19}$$

III. The Scale Operator

The fundamental quantity that characterizes a representation is the operator. We take the following operator for scale

$$\mathcal{C} = \tfrac{1}{2}(\mathcal{T}\mathcal{W} + \mathcal{W}\mathcal{T}) \tag{3.1}$$

and we use the lower case c to represent scale values. The following basic relations show that it scales or compresses functions of time, $f(t)$, and frequency $F(\omega)$,

$$\begin{aligned} e^{j\sigma\mathcal{C}}f(t) &= e^{\sigma/2}f(e^{\sigma}t); \\ e^{j\sigma\mathcal{C}}F(\omega) &= e^{-\sigma/2}F(e^{-\sigma}\omega). \end{aligned} \tag{3.2}$$

Also,

$$\begin{aligned} e^{j\ln\sigma\mathcal{C}}f(t) &= \sqrt{\sigma}\, f(\sigma t); \\ e^{j\ln\sigma\mathcal{C}}F(\omega) &= \sqrt{1/\sigma}\, F(\omega/\sigma). \end{aligned} \tag{3.3}$$

The operator, $e^{j\sigma\mathcal{C}}$ therefore compresses the independent variable. It may be called the compression operator. The significance of the factor, $e^{\sigma/2}$, is that it preserves normalization and the reason it entered in an automatic way is because the operator, $e^{j\sigma\mathcal{C}}$ is unitary. The compression operator should be contrasted with the translation operators for time and frequency functions,

$$e^{j\tau\mathcal{W}}f(t) = f(t + \tau); \qquad e^{j\theta\mathcal{T}}F(\omega) = F(\omega - \theta). \tag{3.4}$$

The basic property of the compression operator, (3.2) can be proven in a number of ways. A purely algebraic proof will now be given and in Section IV, we present a proof based on the eigenfunctions of the scale operator. First, it is readily established that

$$\mathcal{C}t^n = -j(n + \tfrac{1}{2})t^n \tag{3.5}$$

and by repeated action

$$\mathcal{C}^k t^n = (-j)^k (n + \tfrac{1}{2})^k t^n. \tag{3.6}$$

Now,

$$\begin{aligned} e^{j\sigma\mathcal{C}}t^n &= \sum_{k=0}^{\infty} \frac{(j\sigma)^k}{k!}\mathcal{C}^k t^n \\ &= \sum_{k=0}^{\infty} \frac{(j\sigma)^k}{k!}(-j)^k\left(n + \frac{1}{2}\right)^k t^n = e^{\sigma(n+1/2)}t^n. \end{aligned} \tag{3.7}$$

To obtain the action of $e^{j\sigma\mathcal{C}}$ on an arbitrary function $f(t)$, we expand the function in a power series

$$\begin{aligned} e^{j\sigma\mathcal{C}}f(t) &= e^{j\sigma\mathcal{C}}\sum a_n t^n \\ &= \sum a_n e^{\sigma(n+1/2)}t^n = e^{\sigma/2}f(e^{\sigma}t). \end{aligned} \tag{3.8}$$

The scale operator is adopted from the theory of quantum optics where it is written in terms of the creation and annihilation operators [22]. It is related to the affine group [17]–[19]. Also, Klauder [23] has used it to study path integrals and quantum gravity.[3]

By using (2.8) the scale operator can be written in the following alternate ways

$$\mathcal{C} = \mathcal{T}\mathcal{W} - \tfrac{1}{2}j\mathcal{I} = \mathcal{W}\mathcal{T} + \tfrac{1}{2}j\mathcal{I}, \tag{3.9}$$

where $\mathcal{I}$ is the unit operator.

A. Basic Properties of $\mathcal{C}$

Scale and time do not commute. In fact

$$[\mathcal{T}, \mathcal{C}] = j\mathcal{T}. \tag{3.10}$$

The significance of this is that we can not find a common representation which diagonalizes both the scale and time operators. In addition, it shows that we must have an uncertainty principle as discussed in Section XIV. However, time commutes with the commutator of time and scale

$$[\mathcal{T}, [\mathcal{T}, \mathcal{C}]] = 0. \tag{3.11}$$

but the commutator of time and scale does not commute with scale,

$$[\mathcal{C}, [\mathcal{T}, \mathcal{C}]] = \mathcal{T}. \tag{3.12}$$

There is another interesting relation between the log of the time operator and scale. It is

$$[\ln, \mathcal{T}, \mathcal{C}] = j. \tag{3.13}$$

B. Scale and Frequency

Similar comments can be made for frequency and scale. The relevant commutation relations are

$$[\mathcal{W}, \mathcal{C}] = -j\mathcal{W}; \qquad [\mathcal{W}, [\mathcal{W}, \mathcal{C}]] = 0, \tag{3.14}$$

$$[\mathcal{C}, [\mathcal{W}, \mathcal{C}]] = -\mathcal{W}; \qquad [\ln \mathcal{W}, \mathcal{C}] = -j. \tag{3.15}$$

[3] The author would like to thank the referee for bringing [23] to his attention.

IV. The Scale Eigenvalue Problem and the Scale Transform

To obtain the representation of the scale operator we have to solve the eigenvalue problem

$$\mathcal{C}\gamma(c, t) = c\gamma(c, t), \tag{4.1}$$

where we shall use the notation $\gamma(c, t)$ for the scale eigenfunctions. In the time domain the eigenvalue equation is

$$\frac{1}{2j}\left(\frac{d}{dt}t + t\frac{d}{dt}\right)\gamma(c, t) = c\gamma(c, t). \tag{4.2}$$

The eigenfunctions, normalized to a delta function, are readily obtained

$$\gamma(c, t) = \frac{1}{\sqrt{2\pi}}\frac{e^{jc\ln t}}{\sqrt{t}} \qquad t \geq 0. \tag{4.3}$$

They satisfy the completeness relations[4]

$$\int_0^\infty \gamma^*(c', t)\gamma(c, t)\, dt = \delta(c - c'), \tag{4.4}$$

$$\int \gamma^*(c, t')\gamma(c, t)\, dc = \delta(t - t'); \qquad t, t' \geq 0 \tag{4.5}$$

Accordingly, for a signal which goes from minus infinity to plus infinity the positive and negative time parts have to be treated separately.[5] Since we have completeness and orthogonality any time function can be expanded as

$$f(t) = \int D(c)\gamma(c, t)\, dc = \frac{1}{\sqrt{2\pi}}\int D(c)\frac{e^{jc\ln t}}{\sqrt{t}}\, dc \qquad t \geq 0 \tag{4.6}$$

The inverse transformation is

$$D(c) = \int_0^\infty f(t)\gamma^*(c, t)\, dt = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(t)\frac{e^{-jc\ln t}}{\sqrt{t}}\, dt \tag{4.7}$$

We can write $D(c)$ as

$$D(c) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(t)t^{-jc-1/2}\, dt \tag{4.8}$$

which shows that it is the Mellin transform with the complex argument $-jc + 1/2$. We note that for these values of the Mellin transform a simple inversion formulas exist as given by (4.6) and (4.7). Also, we emphasize that the value of c must be taken as real and range from minus infinity to infinity. The reality comes about because they are the eigenvalues of a Hermitian operator. Of course, it has been previously realized that the Mellin transform "scales" and a considerable body of literature exists which relates the scaling of functions with the Mellin Transform. See [13]–[16] and references therein. Also, in relation to joint distributions the original intuition of Marinovich [15], [16] J. Bertrand and P. Bertrand [17], [19], and Altes [14] in deriving specific joint representation was based on the scaling property of the Mellin Transform. See Section XV for further discussion in regards to joint representations.

A. The Scale Transform

We shall call $D(c)$ the scale transform; $|D(c)|^2$ the energy density scale spectrum in analogy to the Fourier case where we have energy density spectrum. We now list some of the basic properties of the scale transform. These properties may be readily derived or one can specialize the known results for the Mellin transform [24] for the case where the index is $jc + 1/2$. First, it is clear that the scale transform is linear and the scale transform of the complex conjugate is $D^*(-c)$. Other important properties are

$$\text{Time scaling: } \sqrt{a}\, f(at) \Leftrightarrow e^{jc\ln a}D(c). \tag{4.9}$$

That is, $f(t)$ and $\sqrt{a}\, f(at)$ have the same energy density scale spectrum, $|D(c)|^2$. This a basic result which shows that the scaling of any time function leaves the energy density scale spectrum unchanged. This is the analog to the fact that the translation of a function leaves the (frequency) energy density spectrum unchanged.

$$\text{Multiplication by time: } t^n f(n) \Leftrightarrow D(c + nj), \tag{4.10}$$

$$\text{Time differentiation: } \frac{df(t)}{dt} \Leftrightarrow \frac{jc + \frac{1}{2}}{\sqrt{2\pi}}\int_0^\infty \frac{f(t)}{t}\frac{e^{-jc\ln t}}{\sqrt{t}}\, dt. \tag{4.11}$$

That is, the scale transform of the derivative of a function is $jc + \frac{1}{2}$ times the scale transform of $f(t)/t$.

B. Properties of the Scale Eigenfunctions

We list here some simple algebraic properties of the eigenfunctions which are useful in calculations

$$\gamma(c, tt') = \gamma(c, t)\gamma(c, t'); \quad \gamma(c, t/t') = \gamma(c, t)\gamma^*(c, t'), \tag{4.12}$$

$$\gamma(c + c', t) = \gamma(c, t)\gamma(c', t); \quad e^{j\sigma c}\gamma(c, t) = e^{\sigma/2}\gamma(c, e^\sigma t) \tag{4.13}$$

$$\frac{\partial\gamma}{\partial c} = j\ln t\gamma(c, t); \quad \frac{\partial\gamma}{\partial t} = \frac{jc - 1/2}{t}\gamma(c, t). \tag{4.14}$$

[4] Integrals over the scale value c go from $-\infty$ to ∞. In establishing the completeness relations one has to uses the fact that $\delta(\ln(x/a)) = a\delta(x - a)$.

[5] The author would like to thank J. Bertrand and P. Bertrand for conversations and insights regarding these issues.

C. Alternate Proof of the Action of the Compression Operator

We now give an alternate proof of the compression operator property, (3.2). It uses the fact that the scale eigenfunctions have the property $e^{j\sigma c}\gamma(c, t) = e^{\sigma/2}\gamma(c, e^{\sigma}t)$. Consider,

$$\begin{aligned} e^{j\sigma \mathfrak{C}} f(t) &= e^{j\sigma \mathfrak{C}} \int D(c)\gamma(c, t)\, dc \\ &= \int D(c) e^{j\sigma c}\gamma(c, t)\, dc \\ &= \int D(c) e^{\sigma/2}\gamma(c, e^{\sigma}t)\, dc = e^{\sigma/2} f(e^{\sigma}t). \end{aligned} \tag{4.15}$$

D. Time and Frequency in the Scale Representation

The fundamental idea of the operator method is that it allows us to calculate expectation directly in any representation. In particular for a time function $g(t)$ we can calculate its expectation value either in time representation of the scale representation, per the general prescription,

$$\langle g(t) \rangle = \int_0^{\infty} g(t)\,|f(t)|^2\, dt = \int D^*(c) g(\mathfrak{I}_c) D(c)\, dc. \tag{4.16}$$

where $\mathfrak{I}_c$ is the time operator in the scale representation. Similar remarks hold for the frequency operator, $\mathfrak{W}_c$. These operators are explicitly given by

$$\begin{aligned} \mathfrak{I}_c &= \exp\left(j\frac{d}{dc}\right); \\ \mathfrak{W}_c &= (c - j)\exp\left(-j\frac{d}{dc}\right). \end{aligned} \tag{4.17}$$

That these are the appropriate operators can be directly verified or by using the approach for transforming operators [25], [26]. Their action is

$$\begin{aligned} \mathfrak{I}_c D(c) &= D(c + j); \\ \mathfrak{W}_c D(c) &= (c - j) D(c - j) \end{aligned} \tag{4.18}$$

where $D(c)$ is an arbitrary scale function.

E. Relation with Other Transforms

The relationship between the Mellin transform and the Laplace and Fourier transform is well known [24]. Since we are dealing with the Mellin transform of a complex argument the appropriate relation is to the Fourier transform. In particular if we define a signal $f_l(t)$ by

$$f_l(t) = \frac{1}{\sqrt{t}} f(\ln t). \tag{4.19}$$

then

$$\begin{aligned} D_l(c) &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f_l(t) \frac{e^{-jc\ln t}}{\sqrt{t}}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(\ln t) \frac{e^{-jc\ln t}}{t}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-jct}\, dt \end{aligned} \tag{4.20}$$

which is the Fourier transform of $f(t)$. That is $F(c) = D_l(c)$. Inversely, one can consider the Fourier transform to be the scale transform of the function $f(e^t)e^{t/2}$. That is if we define

$$f_k(t) = f(e^t)e^{t/2} \tag{4.21}$$

then the Fourier transform of $f_k(t)$ is

$$\begin{aligned} F_k(c) &= \frac{1}{\sqrt{2\pi}} \int f_k(t) e^{-jct}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(e^t) e^{t/2} e^{-jct}\, dt \\ &= = \frac{1}{\sqrt{2\pi}} \int f(t) \frac{e^{-jc\ln t}}{\sqrt{t}}\, dt = D(c). \end{aligned} \tag{4.22}$$

Recently Laine [27] and Baraniuk and Jones [28], [29] have studied interesting class of transforms of which the scale transform is a particular case. Laine studied transform kernels of the form

$$u(t, a) = \frac{1}{\sqrt{2\pi}} \sqrt{g'(t)}\, e^{jag(t)} \tag{4.23}$$

where g is the "generating function." He has shown that these the resulting transform have interesting properties and has discussed these properties in relation to wavelets. The scale transform is a special case where a is taken to be scale c and $g(t) = \ln t$. Baraniuk and Jones [28], [29] studies a class of warping functions or coordinate transformations given by

$$s(t) \rightarrow \sqrt{w'(t)}\, s(w(t))$$

where $w(t)$ is the warping function. They have shown how coordinate transformations may be used to obtain different representations and in particular to study scale. Also, they have applied it to the study of joint representations. See Section XV.

F. Convolution and Correlation

Recently, a general approach for obtaining convolution and correlation theorems has been developed for arbitrary operators [2]. When these are applied to the scale operator one obtains the following: Suppose $D_1(c)$ and $D_2(c)$ are the scale transforms of the signals $f_1(t)$ and $f_2(t)$ respectively then the signal whose transform is $f(t) \Leftrightarrow$

$D_1(c)D_2(c)$ is given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\tau} f_1(1/\tau) f_2(t\tau)\, d\tau$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\tau} f_1(\tau) f_2(t/\tau)\, d\tau. \tag{4.24}$$

Also, the signal whose transform is $D_1^*(c)D_2(c)$ is

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f_1^*(\tau) f_2(t\tau)\, d\tau. \tag{4.25}$$

Of course, these particular cases may be derived in many ways, however the advantage of using the method is that one can derive them for arbitrary operators and does not have to use the properties of a specific case [2]. Also, the above relations can be obtained by specializing the convolution theorems for the Mellin transform [24].

V. Characteristic Function

The characteristic function for scale is

$$M(\sigma) = \int_0^\infty f^*(t) e^{j\sigma\mathcal{C}} f(t)\, dt \tag{5.1}$$
$$= \int_0^\infty f^*(t) e^{\sigma/2} f(e^\sigma t)\, dt$$
$$= \int_0^\infty f^*(e^{-\sigma/2} t) f(e^{\sigma/2} t)\, dt. \tag{5.2}$$

In terms of the spectrum, it is given by

$$M(\sigma) = \int F^*(e^{\sigma/2}\omega) F(e^{-\sigma/2}\omega)\, d\omega. \tag{5.3}$$

The distribution can be obtained by Fourier inversion,

$$P(c) = \frac{1}{2\pi} \int M(\sigma) e^{-j\sigma c}\, d\sigma = |D(c)|^2 \tag{5.4}$$

The distribution of scale has been derived by a number of authors using different methods [14]–[19].

Total Energy Conservation: The total energy, E, of a signal is obtained by integrating over all time the energy density, $|f(t)|^2$, and it should also be obtainable by integrating $|D(c)|^2$ over all scale. It is straightforward to shows that

$$\int_0^\infty |f(t)|\, dt = \int |D(c)|^2\, dc. \tag{5.5}$$

This shows that the total energy is preserved in the transformation. This is the analog to Parcival's theorem for the frequency case.

VI. Mean Scale and Bandwidth

The mean scale is given by

$$\langle c \rangle = \int c |D(c)|^2\, dc. \tag{6.1}$$

This can be evaluated and written in a very interesting form if we write the signal in terms of its phase and amplitude

$$f(t) = A(t) e^{j\varphi(t)}. \tag{6.2}$$

If we substitute the scale transform in (6.1) we will obtain, after some algebra, [6.6] below. However, there is a very easy alternate way to derive this relation if we use the operator method, (2.10). Accordingly, we have

$$\langle c \rangle = \int_0^\infty f^*(t) \mathcal{C} f(t)\, dt. \tag{6.3}$$

But,

$$\mathcal{C} f(t) = -jtf'(t) - \tfrac{1}{2} jf(t)$$
$$= [t\varphi'(t) - j(tA'/A + \tfrac{1}{2})] f. \tag{6.4}$$

Therefore

$$\langle c \rangle = \int_0^\infty t\varphi'(t) A^2(t)\, dt - j \cdot \int_0^\infty (tA'/A + \tfrac{1}{2}) A^2(t)\, dt \tag{6.5}$$

But, the integrand of the second integral is a perfect differential, $(tA'/A + \frac{1}{2})A^2(t) = \frac{1}{2}(d/dt)tA^2$, and hence integrates to zero if we assume that the signal amplitude is zero at zero and infinity. Hence,

$$\langle c \rangle = \int_0^\infty t\varphi'(t) A^2(t)\, dt. \tag{6.6}$$

The scale bandwidth, B, defined by

$$B^2 = \int (c - \langle c \rangle)^2 |D(c)|^2\, dc, \tag{6.7}$$

can be evaluated by direct substitution; however, as above, it is most easily evaluated by calculating

$$B^2 = \int_0^\infty f^*(t) (\mathcal{C} - \langle c \rangle)^2 f(t)\, dt$$
$$= \int |(\mathcal{C} - \langle c \rangle) f(t)|^2\, dt. \tag{6.8}$$

This evaluates to

$$B^2 = \int_0^\infty \left(t \frac{A'(t)}{A(t)} + \frac{1}{2} \right)^2 A^2(t)\, dt + \int_0^\infty (t\varphi'(t) - \langle c \rangle)^2 A^2(t)\, dt. \tag{6.9}$$

Mean scale and bandwidth can also be expressed in terms of the spectrum. If we write the Fourier transform in terms of its phase and amplitude, $F(\omega) = |F(\omega)| e^{j\psi(\omega)}$, then

$$\langle c \rangle = -\int_{-\infty}^\infty \omega\psi'(\omega) |F(\omega)|^2\, d\omega. \tag{6.10}$$

$$B^2 = \int_0^\infty \left(\omega \frac{|F(\omega)|'}{|F(\omega)|} + \frac{1}{2}\right)^2 |F(\omega|^2 \, d\omega + \int_0^\infty (\omega\psi'(\omega) + \langle c \rangle)^2 |F(\omega)|^2 \, d\omega. \quad (6.11)$$

The Covariance of a Signal: The covariance of two variables is a global average indicating how strongly two quantities are related to each other. For the case of time and frequency we can define the covariance by [10]

$$\text{Cov}\,(t\omega) = \langle t\varphi'(t) \rangle - \langle t \rangle \langle \omega \rangle \quad (6.12)$$

where

$$\langle t\varphi'(t) \rangle = \int t\varphi'(t)\,|f(t)|^2. \quad (6.13)$$

The reason for doing so is that if we consider φ' to be the instantaneous frequency then $\langle t\varphi'(t) \rangle$ is the first mixed moment of the two quantities, time and frequency. However we can place ourselves in the frequency domain. There time is given by the group delay which is $-\omega\psi(\omega)$ and frequency just ω. Therefore, it is also reasonable to then define the covariance by

$$\text{Cov}\,(t\omega) = -\langle \omega\psi'(\omega) \rangle - \langle t \rangle \langle \omega \rangle \quad (6.14)$$

where

$$\langle \omega\psi'(\omega) \rangle = \int \omega\psi'(\omega)\,|F(\omega)|^2 \, d\omega. \quad (6.15)$$

It is not obvious that these definitions are equivalent. But they are because

$$\langle \mathcal{C} \rangle = \int t\varphi'(t)\,|f(t)|^2 \, dt = -\int \omega\psi'(\omega)\,|F(\omega)|^2 \, d\omega. \quad (6.16)$$

The concept of the covariance for a signal will be discussed in another publication [30]. We mention it because we note that the covariance involves $\langle t\varphi'(t) \rangle$ and/or $\langle \omega\psi(\omega) \rangle$ which are the mean scale. Indeed, the proof of (6.16) involves the derivations we have given above. We also point out that (6.16) is inherently interesting because it connects the phases and amplitude of a signal and its Fourier transform. However we point out that in (6.13) the limits of integration go from $-\infty$ to ∞.

VII. The Short-Time Scale Transform

Suppose we want to study local values of scale such as instantaneous scale. As in the Fourier case we focus on a particular time by windowing the signal around that time. In particular, we window the signal with a windowing function, $h(t)$, and define the short-time scale transform [6] by

$$D_t(c) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(\tau)h(\tau - t) \frac{e^{-jc\ln\tau}}{\sqrt{\tau}} \, d\tau \quad (7.1)$$

The joint distribution of time and scale is then given by

$$P(t, c) = |D_t(c)|^2. \quad (7.2)$$

The time marginal is

$$P(t) = \int P(t, c) \, dc = \int |D_t(t)|^2 \, dc = \int_0^\infty A^2(\tau)A_h^2(\tau - t) \, d\tau. \quad (7.3)$$

where we have written the windowing function in terms of its amplitude and phase $h(t) = A_h(t)\,e^{j\varphi_h(t)}$.

VIII. Instantaneous Scale

We now give two different approaches for the concept of instantaneous scale. The first is based on the short time scale transform and the second is based on the scale bandwidth equation. Both methods give the same answer. For a fuller discussion of these approaches and other approaches see reference [6].

A. The Short-Time Scale Transforms

Using the short time-scale transform the average scale for a given time is,

$$\langle c \rangle_t = \frac{1}{P(t)} \int cP(t, c) \, dc = \frac{1}{P(t)} \int c\,|D_t(c)|^2 \, dc. \quad (8.1)$$

This may be thought of as the conditional average of scale for a given time. We now give a simple method for evaluating this quantity. Note that (8.1) has the same form as (6.1) with D_t substituted for D. Therefore, we can write

$$\langle c \rangle_t = \frac{1}{P(t)} \int s_h(\tau)\,\mathcal{C}(\tau)\,s_h(\tau) \, d\tau \quad (8.2)$$

where

$$s_h(\tau) = f(\tau)h(\tau - t) = A(t)A_h(\tau - t) \exp\,[j\varphi(t) + j\varphi_h(\tau - t)]. \quad (8.3)$$

The evaluation is immediate

$$\langle c \rangle_t = \frac{1}{P(t)} \int_0^\infty A^2(\tau)A_h^2(\tau - t) \cdot \tau\{\varphi'(\tau) + \varphi_h'(\tau - t)\} \, d\tau. \quad (8.4)$$

Equation (8.4) can be considered an estimate for scale at a given time. As our interest is to obtain instantaneous scale we progressively narrow the window in time which will give us a progressively better estimate. In particular, we narrow the window in such a way that $A_h^2(t)$ approaches a delta function, $A_h^2(t) \sim \delta(t)$. Also, we consider real windows so as not to introduce any phase factors. In that limit, we find the marginal, as given by (7.3), $P(t) \sim A^2(t)$. Now the numerator of (8.4) goes as $= A^2(t)\,t\varphi'(t)$, and hence $\langle c \rangle_t \sim t\varphi'(t)$. We shall call that the instantaneous scale, c_t,

$$c_t = t\varphi'(t). \quad (8.5)$$

The general relation of a conditional mean with the global mean is that the global mean is the average of the conditional mean. For scale we expect that

$$\langle c \rangle = \int_0^\infty c_t |f(t)|^2 \, dt \tag{8.6}$$

and that is indeed the case as can be seen by referring to (6.6).

Similar to instantaneous scale in time we may ask for the instantaneous scale in frequency, c_ω. That is, the value of scale for a given frequency. The identical arguments yields $c_\omega = -\omega\psi'(\omega)$.

B. Bandwidth Equation

In Section VI we derived an expression for the scale bandwidth, (6.9). Our aim here is to show that it has a physical interpretation and to use it to derive the instantaneous scale bandwidth. In addition we will give an alternative derivation of instantaneous scale.

We write (6.9) here for convenience,

$$B^2 = \int_0^\infty \left(t \frac{A'(t)}{A(t)} + \frac{1}{2}\right)^2 A^2(t) \, dt + \int_0^\infty (t\varphi'(t) - \langle c \rangle)^2 A^2(t) \, dt. \tag{8.7}$$

To understand the meaning of this equation consider first the general problem of two variables, x and y, with density $P(x, y)$. It is straightforward to show [31] that the standard deviation of one variable, say y, is given by

$$\sigma_y^2 = \int \sigma_{y|x}^2 P(x) \, dx + \int (\langle y \rangle_x - \langle y \rangle)^2 P(x) \, dx \tag{8.8}$$

where $\langle y \rangle_x$ is the conditional average and $\sigma_{y|x}^2$ is the conditional standard deviation. $P(x)$ is the density in x. Now compare this equation with our case, (8.7). It is suggestive to take, for instantaneous scale, $\langle c \rangle_t = t\varphi'(t)$ and to define the local standard deviation for scale (instantaneous scale bandwidth) by

$$\sigma_{c|t}^2 = \left(t \frac{A'(t)}{A(t)} + \frac{1}{2}\right)^2. \tag{8.9}$$

This is analogous to the concept of instantaneous bandwidth developed by Cohen and Lee [31].

C. Frequency

Using (6.11) we have a similar interpretation for scale at a particular frequency and its conditional standard deviation,

$$\langle c \rangle_\omega = -\omega\psi'(\omega); \qquad \sigma_{c|\omega}^2 = \left(\omega \frac{|F(\omega)|'}{|F(\omega)|} + \frac{1}{2}\right)^2. \tag{8.10}$$

IX. Average Time and Duration in the Scale Representation, and Scale Group Delay

The mean time of a signal is defined by

$$\langle t \rangle = \int t |f(t)|^2 \, dt. \tag{9.1}$$

It is interesting to express this in the scale representation. To do so one could substitute $f(t)$ in terms of $D(c)$ into (9.1) and do the algebra to derive (9.5). However, since we know the time operator, $\mathfrak{I}_c$, in the scale representation we can calculate the mean time by way of

$$\langle t \rangle = \int D^*(c) \mathfrak{I}_c D(c) \, dc. \tag{9.2}$$

Using (4.17) for the time operator and using (4.18) for its action on an arbitrary scale function we have

$$\langle t \rangle = \int D^*(c) D(c + j) \, dc = \int \frac{D(c + j)}{D(c)} |D(c)|^2 \, dc. \tag{9.3}$$

We now break up $D(c + j)/D(c)$ into its real and imaginary parts

$$\frac{D(c + j)}{D(c)} = \left(\frac{D(c + j)}{D(c)}\right)_R + j\left(\frac{D(c + j)}{D(c)}\right)_I. \tag{9.4}$$

Since we know the time operator is Hermitian its average must be real and hence the imaginary part must integrate to zero. Therefore,

$$\langle t \rangle = \int \left(\frac{D(c + j)}{D(c)}\right)_R |D(c)|^2 \, dc. \tag{9.5}$$

We point out that

$$D(c + j) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \sqrt{t} f(t) e^{-jc \ln t} \, dt. \tag{9.6}$$

which is the scale transform of $tf(t)$.

Duration: The duration is

$$T^2 = \int_0^\infty (t - \langle t \rangle^2 |f(t)|^2 \, dt = \int D^*(c) (\mathfrak{I}_c - \langle t \rangle)^2 D(c) \, dc. \tag{9.7}$$

Manipulation of this expression leads to

$$T^2 = \int \left(\frac{D(c + j)}{D(c)}\right)_I^2 |D(c)|^2 \, dc + \int \left[\left(\frac{D(c + j)}{D(c)}\right)_R - \langle t \rangle\right]^2 |D(c)|^2 \, dc. \tag{9.8}$$

X. Average Frequency and Bandwidth in the Scale Representation

To obtain the mean frequency, in the scale representation we again use the operator method,

$$\langle \omega \rangle = \int \omega |F(\omega)|^2 \, d\omega = \int D^*(c) \mathcal{W}_c D(c) \, dc \tag{10.1}$$

$$= \int D^*(c)(c - j) D(c - j) \, dc$$

$$= \int c \left(\frac{D(c - j)}{D(c)} \right)_R |D(c)|^2 \, dc + \int \left(\frac{D(c - j)}{D(c)} \right)_I |D(c)|^2 \, dc \tag{10.2}$$

which is the mean frequency expressed in the scale representation.

For the frequency bandwidth, B_ω, we have

$$B_\omega^2 = \int (\omega - \langle \omega \rangle)^2 |F(\omega)|^2 \, d\omega$$

$$= \int D^*(c)(\mathcal{W}_c - \langle \omega \rangle)^2 D(c) \, dc \tag{10.3}$$

$$= \int \left[c \left(\frac{D(c - j)}{D(c)} \right)_I - \left(\frac{D(c - j)}{D(c)} \right)_R \right]^2 |D(c)|^2 \, dc \tag{10.4}$$

$$+ \int \left[c \left(\frac{D(c - j)}{D(c)} \right)_R + \left(\frac{D(c - j)}{D(c)} \right)_I - \langle \omega \rangle \right]^2 |D(c)|^2 \, dc. \tag{10.5}$$

Group Scale Delay: We introduce the concept of scale group delay in analogy with group delay for frequency. Group delay is the average time for a given frequency. Similarly we define the scale group delay to be the average time for a given scale. That is, it is the conditional time for a given scale, $\langle t \rangle_c$. We expect this quantity to be connected to the mean time by

$$\langle t \rangle = \int \langle t \rangle_c |D(c)|^2 \, dc. \tag{10.6}$$

Comparing (10.2) with (8.8) it is reasonable to take

$$\langle t \rangle_c = \left(\frac{D(c + j)}{D(c)} \right)_R; \qquad \sigma_{t|c}^2 = \left(\frac{D(c + j)}{D(c)} \right)_I^2. \tag{10.7}$$

An alternative derivation of these relations has been given elsewhere [6].

We also define the group scale delay for a given frequency, $\langle \omega \rangle_c$. The identical arguments as above lead to

$$\langle \omega \rangle_c = c \left(\frac{D(c - j)}{D(c)} \right)_R - \left(\frac{D(c - j)}{D(c)} \right)_I \tag{10.8}$$

$$\sigma_{\omega|c}^2 = \left[c \left(\frac{D(c - j)}{D(c)} \right)_I - \left(\frac{D(c - j)}{D(c)} \right)_R \right]^2. \tag{10.9}$$

XI. Analytic Scale Signal and Filtering

In the frequency representation the analytic signal is defined for a number of reasons [10], [32], [33]. First, it allows one to define unambiguously the amplitude and phase of a signal. Second, it eliminates the negative frequencies. We obtain the analytic *scale* signal, $z(t)$, by zeroing out the negative scale values of the signal,

$$z(t) = 2 \frac{1}{\sqrt{2\pi}} \int_0^\infty D(c) \frac{e^{jc \ln t}}{\sqrt{t}} \, dc. \tag{11.1}$$

The factor of two assures that the real part of $z(t)$ is the original signal. Using the same method as to obtain the analytic signal we have

$$z(t) = f(t) + \frac{j}{\pi} \int_0^\infty \frac{f(t')}{\sqrt{tt'} \ln (t/t')} \, dt'$$

$$= f(t) + \frac{j}{\pi \sqrt{t}} \int_0^\infty \frac{f(e^\tau)}{\ln t - \tau} e^{\tau/2} \, d\tau. \tag{11.2}$$

The above consideration can be looked upon as filtering out the negative scale values. More generally the concept of filtering in the frequency domain arises for the following reasons. Suppose we have a signal and we want to eliminate certain frequencies. We accomplish that by multiplying the Fourier transform by a function which zeros out those frequencies. This produces a Fourier transform without the frequences in question. The new signal is obtained by inverting the modified Fourier transform. Similarly, if we have a signal, $f(t)$, with corresponding scale transform, $F(c)$, and want to filter out some values of scale, we multiply $F(c)$ by a function $H(c)$ to obtain a

$$G(c) = F(c) H(c). \tag{11.3}$$

The time function which produces this filtered transform is

$$y(t) = \int G(c) \gamma(c, t) \, dc = \int F(c) H(c) \gamma(c, t) \, dc. \tag{11.4}$$

But we already know the answer from (4.24). It is

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{r} f(1/\tau) h(t\tau) \, d\tau \tag{11.5}$$

where

$$h(t) = \frac{1}{\sqrt{2\pi}} \int H(c) \frac{e^{jc \ln t}}{\sqrt{t}} \, dc; \qquad t \geq 0. \tag{11.6}$$

XII. Linear Scale-Invariant Systems

Our aim is to formulate linear scale-invariant systems in the same manner as is done for frequency where one considers linear shift-invariant systems. We have given a general method to obtain system functions for arbitrary operators [2]. We now specialize that approach to the case of scale. As in the standard case we take the output signal,

$g(t)$, to be related to the input signal, $f(t)$, by

$$g(t) = \mathfrak{L}f(t), \tag{12.1}$$

where $\mathfrak{L}$ is the system operator. We define the impulse response function, $h(t)$, for scale-invariant systems by

$$\mathfrak{L}\{\delta(t-1)\} = h(t). \tag{12.2}$$

A system is said to be scale-invariant if for any number a, the following condition holds

$$\mathfrak{L}\{\delta(t-1/a)\} = ah(at). \tag{12.3}$$

The impulse response function characterizes the system in the sense that the output function can be obtained from the input function and the impulse response function. Consider the output $g(t) = \mathfrak{L}f(t)$. One may show that [2]

$$\begin{aligned} g(t) &= \int \frac{1}{t'} f(t') h(t/t')\, dt' \\ &= \int \frac{1}{t'} f(t/t') h(t')\, dt', \end{aligned} \tag{12.4}$$

Therefore a knowledge of the system response function determines the output.

XIII. Scale Energy Density and Correlation

For the case of frequency the energy density spectrum can be expressed as the Fourier transform of the autocorrelation function.[6] We generalize this idea to the scale energy density spectrum, $|D(c)|^2$. We present two generalizations.

A. Relation One

$$|D(c|^2 = \int_0^\infty K(\tau)\gamma^*(c, \tau)\, d\tau \tag{13.1}$$

where

$$K(\tau) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f^*(t) f(\tau t)\, dt. \tag{13.2}$$

There are two alternative ways of writing $K(t)$

$$K(\tau) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f^*(t) \frac{e^{j \ln \tau \mathcal{C}}}{\sqrt{\tau}} f(t)\, dt. \tag{13.3}$$

$$\begin{aligned} K(\tau) = \int_0^\infty \int_0^\infty \int f^*(t)\gamma^*(c, t')\gamma(c, t)\gamma(c, \tau) \\ \cdot f(t')\, dc\, dt'\, dt. \end{aligned} \tag{13.4}$$

B. Relation Two

$$|D(c)|^2 = \int M(\tau) e^{-j\tau c}\, d\tau \tag{13.5}$$

with

$$\begin{aligned} M(\tau) &= \int_0^\infty f^*(t) e^{\tau/2} f(e^\tau t)\, dt \\ &= \int_0^\infty f^*(t) e^{j\tau\mathcal{C}} f(t)\, dt. \end{aligned} \tag{13.6}$$

These equations can be considered as generalizations of the Wiener–Khinchin theorem for deterministic signals for the property scale. Their proofs are straightforward. In proving the above relations it is useful to note that $\delta(\ln(z/a)) = a\delta(x-a)$.

XIV. Uncertainty Principles for Scale

There is an uncertainty principle for any two quantities which are represented by operators which do not commute [21]. For any two quantities a and b represented by the respective operators, $\mathcal{A}$ and $\mathcal{B}$, the uncertainty principle is

$$\Delta a\, \Delta b \geq \tfrac{1}{2}|\langle[\mathcal{A}, \mathcal{B}]\rangle|, \tag{14.1}$$

where $(\Delta a)^2$ is the standard deviation, defined in the usual way, $(\Delta a)^2 = \langle a^2\rangle - \langle a\rangle^2$, and similarly for $(\Delta b)^2$. We note that since the operators are Hermitian one can calculate $(\Delta a)^2$ from

$$(\Delta a)^2 = \int f^*(t)(\mathcal{A} - \langle a\rangle)^2 f(t)\, dt \tag{14.2}$$

$$= \int |(\mathcal{A} - \langle a\rangle) f(t)|^2\, dt. \tag{14.3}$$

Furthermore, to obtain the signal which minimizes the uncertainty product one has to solve

$$(\mathcal{B} - \langle b\rangle) f(t) = \lambda(\mathcal{A} - \langle a\rangle) f(t) \tag{14.4}$$

where $\lambda = \langle[\mathcal{A}, \mathcal{B}]\rangle / 2(\Delta\mathcal{A})^2)$. Note that in general λ may be complex since $[\mathcal{A}, \mathcal{B}]$ is not Hermitian.

A. Uncertainty Principle: Time and Scale

The commutator for time and scale is given by $[\mathfrak{I}, \mathcal{C}] = j\mathfrak{I}$ and we immediately have $\Delta t\, \Delta c > \frac{1}{2}|\langle[\mathfrak{I}, \mathcal{C}]\rangle|$ or

$$\Delta t\, \Delta c \geq \tfrac{1}{2}|\langle t\rangle|, \tag{14.5}$$

where $\langle t\rangle$ is the mean time.

The equation which gives the minimum uncertainty product is

$$\left(\frac{1}{j}\left[t\frac{d}{dt} + \frac{1}{2}\right] - \langle c\rangle\right) f(t) = \lambda(t - \langle t\rangle) f(t), \tag{14.6}$$

where

$$\lambda = \frac{\langle[\mathfrak{I}, \mathcal{C}]\rangle}{2(\Delta t)^2} = j\frac{\langle t\rangle}{2(\Delta t)^2} \tag{14.7}$$

and where Δt is the duration and $\langle c\rangle$ is the mean scale. The solution of (14.6) is

$$f(t) = kt^{\alpha_2} \exp[-\alpha_1 t + j\langle c\rangle \ln(t/\langle t\rangle)] \tag{14.8}$$

[6] We deal here with deterministic signals in which case when we say autocorrelation function we properly mean the deterministic autocorrelation function. The case of random functions is discussed in [2] and application to random scale functions will be discussed in a future paper.

where k is a normalizing constant and

$$\alpha_1 = \frac{\langle t \rangle}{2(\Delta t)^2}; \qquad \alpha_2 = \frac{1}{2}\left(\frac{\langle t \rangle^2}{(\Delta t)^2} - 1\right). \quad (14.9)$$

For the case of the position operator this minimum uncertainty function has been given by Klauder [23].

B. Uncertainty Principle: Frequency and Scale

Using $[\mathcal{W}, \mathcal{C}] = -j\mathcal{W}$, we have that

$$\Delta\omega\, \Delta c \geq \tfrac{1}{2}|\langle[\mathcal{W}, \mathcal{C}]\rangle| = \tfrac{1}{2}|\langle\omega\rangle| \quad (14.10)$$

where $\langle\omega\rangle$ is the mean frequency. The signal that minimizes the frequency-scale uncertainty is

$$f(k) = k(t - \lambda)^{j\langle c\rangle - 1/2 - \lambda\langle\omega\rangle}. \quad (14.11)$$

For the case of log ω and scale the identical solution holds with time being replaced by frequency and the time function by a frequency function.

C. Uncertainty Principle for Log Time and Scale

Using the commutator relation between log time and scale $[\ln \mathcal{T}, \mathcal{C}] = j$, we can write the uncertainty principle

$$\Delta \ln t\, \Delta c \geq \tfrac{1}{2}. \quad (14.12)$$

The minimum uncertainty signal is obtained by solving

$$\left(\frac{1}{j}\left[t\frac{d}{dt} + \frac{1}{2}\right] - \langle c\rangle\right) f(t) = \lambda(\ln t - \langle \ln t\rangle) f(t) \quad (14.13)$$

The solution is

$$f(t) = A \exp\left[-\frac{(\ln t - \langle \ln t\rangle)^2}{4(\Delta \ln t)^2} + j(\langle c\rangle - 1/2)\ln t\right]. \quad (14.14)$$

XV. Joint Time-Scale Representations

We now present a general method for obtaining joint distributions involving scale. The method will allow us to obtain an infinite number of distributions and the general class of time-scale representations. The procedure is a generalization of the method developed by the author for the time-frequency case [10] and further generalized to arbitrary variables by Scully and Cohen [11] and Cohen [1], [7]. As special cases, we obtain previously obtained particular distributions involving scale.

The first investigators to consider the concept of joint representations involving scale were Marinovich [15], [16, Bertrand and Bertrand [17]–[19] who obtained particular but very different distributions. Marinovich arguments were based on using the scaling properties of the Mellin transform and Bertrand and Bertrand used group theoretical methods on the affine group. Altes [14], also using the scaling properties of the Mellin transform obtained distributions akin to Marinovich.

Rioul [34], Posch [35] and, Rioul and Flandrin [36], Papandreaou *et al.* [37] and Cohen [5] have presented other approaches and generalization. The group theoretic approach of R. G. Shenoy and T. W. Parks [38] is closely related to the operator approach presented here. A general approach to study joint representations in signal analysis has recently been presented by Baraniuk and Jones [28], [29]. The relationship between that approach and the general approach based on operators described here and in references [1], [7], [11] will be presented in a future publication [39]. Also, we point out that there is a strong similarity between inverse frequency representation and scale. However, they are different and in particular they satisfy different marginals. The relationship between scale, inverse frequency and joint representations have been discussed by Jeong and Williams [40] and Cohen [5]. Also, Flandrin [41], [42] has extended joint scale representations to the case of random signals by defining a scale invariant Wigner Spectrum.

We now describe the characteristic operator function method as applied to scale. The characteristic function of a joint distribution, $P(t, c)$, is

$$M(\theta, \sigma) = \langle e^{j\theta t + j\sigma c}\rangle = \iint e^{j\theta t + j\sigma c} P(t, c)\, dt\, dc \quad (15.1)$$

and the distribution is obtained from the characteristic function by

$$P(t, c) = \frac{1}{4\pi^2}\iint M(\theta, \sigma) e^{-j\theta t - j\sigma c}\, d\theta\, d\sigma. \quad (15.2)$$

The characteristic function is an average, namely the average of $e^{j\theta t + j\sigma c}$. The basic idea is to replace the scalar quantities t and c by their corresponding operators. We therefore define the characteristic function operator $\mathcal{M}(\theta, \sigma; \mathcal{T}, \mathcal{C})$ to be an operator for which the characteristic function can be obtained by averaging

$$M(\theta, \sigma) = \langle \mathcal{M}(\theta, \sigma; \mathcal{T}, \mathcal{C})\rangle = \int_0^\infty f^*(t)\mathcal{M}(\theta, \sigma; \mathcal{T}, \mathcal{C}) f(t)\, dt. \quad (15.3)$$

Once M is obtained the distribution is calculated by (15.2). Because $\mathcal{T}$ and $\mathcal{C}$ are operators there is an ordering problem. One can take, for example

$$\mathcal{M}(\theta, \sigma; \mathcal{T}, \mathcal{C}) = e^{j\theta\mathcal{T} + j\sigma\mathcal{C}}$$

or

$$\mathcal{M}(\theta, \sigma; \mathcal{T}, \mathcal{C}) = e^{j\sigma\mathcal{C}/2} e^{j\theta\mathcal{T}} e^{j\sigma\mathcal{C}/2}, \text{ etc.} \quad (15.4)$$

and indeed there are an infinite number of possible orderings. For each different ordering a different characteristic function is obtained and hence a different distribution. This is the reason why we have an infinite number of distributions [8]–[10] for the time-frequency case; similarly for scale. It is this wide range of possibilities that lead to the general class of time-frequency distributions and will

TABLE I
TIME-(t) SCALE REPRESENTATIONS

#	Ordering	Characteristic Function: $M(\theta, \sigma)$	Distribution: $P(t, c)$
1	$e^{j\sigma\mathfrak{C}/2}e^{j\theta\mathfrak{T}}e^{j\sigma\mathfrak{C}/2}$	$\int_0^\infty f^*(e^{-\sigma/2}t)e^{j\theta t}f(e^{\sigma/2}t)\,dt$	$\frac{1}{2\pi}\int f^*(e^{-\sigma/2}t)e^{-j\sigma c}f(e^{\sigma/2}t)\,d\sigma$
2	$e^{j\theta\mathfrak{T}+j\sigma\mathfrak{C}}$	$\int_0^\infty \exp\,[2j\theta t\,\sinh\,(\sigma/2)/\sigma]\,f^*(e^{-\sigma/2}t)f(e^{\sigma/2}t)\,dt$	$\frac{1}{2\pi}\int \frac{\sigma}{2\sinh(\sigma/2)}e^{-j\sigma c}f^*\left(e^{-\sigma/2}\frac{\sigma t}{2\sinh(\sigma/2)}\right) \cdot f\left(e^{\sigma/2}\frac{\sigma t}{2\sinh(\sigma/2)}\right)d\sigma$
3	$e^{j\theta\mathfrak{T}/2}e^{j\sigma\mathfrak{C}}e^{j\theta\mathfrak{T}/2}$	$\int_0^\infty \exp\,(j\theta t\,\cosh\,(\sigma/2))\,f^*(e^{-\sigma/2}t)f(e^{\sigma/2}t)\,dt$	$\frac{1}{2\pi}\int \frac{1}{\cosh(\sigma/2)}e^{-j\sigma c}f^*\left(e^{-\sigma/2}\frac{t}{\cosh(\sigma/2)}\right) \cdot f\left(e^{\sigma/2}\frac{t}{\cosh(\sigma/2)}\right)d\sigma$
4	$e^{j\sigma\mathfrak{C}}e^{j\theta\mathfrak{T}}$	$\int_0^\infty f^*(e^{-\sigma/2}t)\exp\,(j\theta e^{\sigma/2}t)f(e^{\sigma/2}t)\,dt$	$f(t)\gamma^*(c, t)D^*(c)$
5	$e^{j\theta\mathfrak{T}}e^{j\sigma\mathfrak{C}}$	$\int_0^\infty f^*(e^{-\sigma/2}t)\exp\,(j\theta e^{-\sigma/2}t)f(e^{\sigma/2}t)\,dt$	$f^*(t)\gamma(c, t)D(c)$
6	$\frac{1}{2}(\{4\} + \{5\})$	$\frac{1}{2}\{4\} + \{5\})$	Real part $\{f^*(t)\gamma(c, t)D(c)\}$

also lead to a general class of time-scale distributions as discussed in Section XVII.

We now give a number of time-scale distributions by taking different operator orderings. We list these in Table I. The details of each case can be found in [1]. Here, we give the detailed steps for case one only. The symmetric ordering, case 2 of the Table I, corresponds to the Weyl ordering when the operators obey the commutation rules given by (2.8). To evaluate the characteristic function one must simplify the operator. The simplification is as follows [1], [43]:

$$e^{j\theta\mathfrak{T}+j\sigma\mathfrak{C}} = e^{j\theta\eta\mathfrak{T}}e^{L\sigma c}e^{j\theta\mathfrak{T}} \tag{15.5}$$

where

$$\eta = -\frac{1}{\sigma}\{1 - (1 - \sigma)e^{\sigma}\}. \tag{15.6}$$

We now give the steps leading to case one. The characteristic function is

$$M(\theta, \sigma) = \langle e^{j\sigma\mathfrak{C}/2}e^{j\theta\mathfrak{T}}e^{j\sigma\mathfrak{C}/2}\rangle$$

$$= \int_0^\infty f^*(t)\,e^{j\sigma\mathfrak{C}/2}e^{j\sigma\mathfrak{T}}e^{j\sigma\mathfrak{C}/2}f(t)\,dt \tag{15.7}$$

$$= \int_0^\infty f^*(t)\,e^{j\sigma\mathfrak{C}/2}e^{j\theta t}e^{\sigma/4}f(e^{\sigma/2}t)\,dt \tag{15.8}$$

$$= \int_0^\infty f^*(t)\,\exp\,(j\theta e^{\sigma/2}t)\,e^{\sigma/2}f(e^{\sigma}t)\,dt, \tag{15.9}$$

or

$$M(\theta, \sigma) = \int_0^\infty f^*(e^{-\sigma/2}t)\,e^{j\theta t}f(e^{\sigma/2}t)\,dt. \tag{15.10}$$

The distribution is

$$P(t, c) = \frac{1}{4\pi^2}\iint M_1(\theta, \sigma)\,e^{-j\theta t - j\sigma c}\,d\theta\,d\sigma \tag{15.11}$$

$$= \frac{1}{4\pi^2}\int_0^\infty \iint f^*(e^{-\sigma/2}t')\,e^{j\theta t'} \cdot f(e^{\sigma/2}t')\,e^{-j\theta t - j\sigma c}\,d\theta\,d\sigma\,dt' \tag{15.12}$$

$$= \frac{1}{2\pi}\int_0^\infty \int f^*(e^{-\sigma/2}t')\,\delta(t - t') \cdot f(e^{\sigma/2}t')\,e^{-j\sigma c}\,d\sigma\,dt'. \tag{15.13}$$

Hence,

$$P(t, c) = \frac{1}{2\pi}\int f^*(e^{-\sigma/2}t)\,e^{-j\sigma c}f(e^{\sigma/2}t)\,d\sigma. \tag{15.14}$$

This distribution was derived by Marinovich [16], [17] and Altes [14] by different methods.

These distributions satisfy the marginals of time and scale,

$$\int P(t, c)\,dt = |D(c)|^2;$$

$$\int P(t, c)\,dc = |s(t)|^2. \tag{15.15}$$

XVI. JOINT FREQUENCY-SCALE REPRESENTATIONS

We now find representations of frequency and scale. However we emphasize that we are seeking representations of frequency and *time* scaling. In section XIX, we will address the issue of frequency scaling.

The notation we shall use is $M(\tau, \sigma)$ and $P(\omega, c)$ for the frequency-scale characteristic function and distribu-

TABLE II
FREQUENCY-(t) SCALE REPRESENTATIONS

#	Ordering	Characteristic Function: $M(\tau, \sigma)$	Distribution: $P(\omega, c)$
1	$e^{j\sigma\mathcal{C}} e^{j\tau\mathcal{W}} e^{j\sigma\mathcal{C}/2}$	$\int F^*(e^{\sigma/2}\omega) e^{j\tau\omega} F(e^{-\sigma/2}\omega)\, d\omega$	$\frac{1}{2\pi} \int F^*(e^{-\sigma/2}\omega) e^{j\sigma c} F(e^{\sigma/2}\omega)\, d\sigma$
2	$e^{j\tau\mathcal{W} + j\sigma\mathcal{C}}$	$\int F^*(e^{\sigma/2}\omega) \exp [2j\tau\omega \sinh (\sigma/2)/\sigma] F(e^{-\sigma/2}\omega)\, d\omega$	$\frac{1}{2\pi} \int \frac{\sigma}{2 \sinh (\sigma/2)} e^{j\sigma c} F^*\left(e^{-\sigma/2} \frac{\sigma\omega}{2 \sinh (\sigma/2)}\right) \cdot F\left(e^{\sigma/2} \frac{\omega}{\cosh (\sigma/2)}\right) d\sigma$
3	$e^{j\tau\mathcal{W}/2} e^{j\sigma\mathcal{C}} e^{j\tau\mathcal{W}/2}$	$\int F^*(e^{\sigma/2}\omega) \exp (j\tau\omega \cosh (\sigma/2)) F(e^{-\sigma/2}\omega)\, d\omega$	$\frac{1}{2\pi} \int \frac{1}{\cosh (\sigma/2)} e^{j\sigma c} F^*\left(e^{-\sigma/2} \frac{\sigma\omega}{\cosh (\sigma/2)}\right) \cdot F\left(e^{\sigma/2} \frac{\omega}{\cosh (\sigma/2)}\right) d\sigma$
4	$e^{j\sigma\mathcal{C}} e^{j\tau\mathcal{W}}$	$\int F^*(e^{\sigma/2}\omega) \exp (-j\tau e^{-\sigma/2}\omega) F(e^{-\sigma/2}\omega)\, d\omega$	$\frac{1}{2\pi} F(\omega) \int e^{-\sigma/2} F^*(e^{-\sigma}\omega) e^{j\sigma c}\, d\sigma$
5	$e^{j\tau\mathcal{W}} e^{j\sigma\mathcal{C}}$	$\int F^*(e^{\sigma/2}\omega) \exp (j\tau e^{\sigma/2}\omega) F(e^{-\sigma/2}\omega) d\omega$	$\frac{1}{2\pi} F^*(\omega) \int e^{\sigma/2} F(e^{\sigma}\omega) e^{j\sigma c}\, d\sigma$
6	$\frac{1}{2}(\{4\} + \{5\})$	$\frac{1}{2}(\{4\} + \{5\})$	Real part $\{4\}$

Table 1 and 2 give, respectively, joint time and frequency representation with time scaling [(t) scale].

tion respectively. They are related by

$$M(\tau, \sigma) = \iint e^{j\tau\omega + j\sigma c} P(\omega, c)\, d\omega\, dc. \tag{16.1}$$

$$P(\omega, c) = \frac{1}{4\pi^2} \iint M(\tau, \sigma) e^{-j\tau\omega - j\sigma c}\, dt\, d\sigma. \tag{16.2}$$

The characteristic function will be obtained from the characteristic function operator, $\mathfrak{M}(\tau, \sigma; \mathcal{W}, \mathcal{C})$ by averaging it

$$M(\tau, \sigma) = \langle \mathfrak{M}(\tau, \sigma, \mathcal{W}, \mathcal{C}) \tag{16.3}$$

$$= \int_0^\infty f^*(t) \mathfrak{M}(\tau, \sigma; \mathcal{W}, \mathcal{C}) f(t)\, dt. \tag{16.4}$$

We could repeat the algebra as in the previous section; however there is a simple procedure to evaluate these cases. First we note that we can evaluate $M(\tau, \sigma)$ in the Fourier domain,

$$M(\tau, \sigma) = \int_{-\infty}^{\infty} F^*(\omega) \mathfrak{M}(\tau, \sigma; \mathcal{W}, \mathcal{C}) F(\omega)\, d\omega \tag{16.5}$$

We are assured that evaluating M by way of (16.4) or (16.5) will give the identical results since which representation is used is immaterial in calculating averages by the operator method. Now, note that structurally $\mathfrak{M}(\tau, \sigma; \mathcal{W}, \mathcal{C}) F(\omega)$ is identical to $\mathfrak{M}(\theta, \sigma; \mathfrak{J}, \mathcal{C}) f(t)$ except for the fact that σ goes into $-\sigma$. Therefore the frequency-scale characteristic function will be identical in form to the time-scale characteristic function, but we must change the sign of σ and replace the time function by its Fourier transform. Also, we must change θ to τ. The distribution is the Fourier transform of the characteristic function. Since everything is structurally the same except for the minus sign in σ, we can obtain the frequency-scale distribution from the time scale distribution by replacing c by $-c$, and t by ω and replace the signal with its Fourier transform. We should emphasize that the limits of integration on the frequency variable go from $-\infty$ to ∞ since the spectrum is not constrained. (In Section XIX we will deal with frequency scaling where the spectrum will have only positive values.) Table II lists the corresponding characteristic function operators, characteristic functions and the distributions corresponding to the ordering cases we have considered in Section XV. The frequency-scale distributions satisfy the marginal of frequency and scale,

$$\int P(\omega, c)\, d\omega = |D(c)|^2; \tag{16.6}$$

$$\int P(\omega, c)\, dc = |F(\omega)|^2. \tag{16.7}$$

XVII. GENERAL CLASS OF JOINT SCALE REPRESENTATIONS

The method we present for obtaining the general class of time-scale distributions is a generalization of the method used to obtain the general class of time-frequency distributions [8]–[10]. We start with any particular characteristic function and define a new characteristic function by

$$M_{\text{new}}(\theta, \sigma) = \phi(\theta, \sigma) M(\theta, \sigma) \tag{17.1}$$

where ϕ is the kernel function whose role is identical to the time-frequency case. The general class is then

$$P(t, c) = \frac{1}{4\pi^2} \iint \phi(\theta, \sigma) M(\theta, \sigma) e^{-j\theta t - j\sigma c}\, d\theta\, d\sigma. \tag{17.2}$$

We emphasize that M is any particular characteristic function.

TABLE III
FREQUENCY-(ω) SCALE REPRESENTATIONS

#	Ordering	Characteristic Function: $M(\tau, \sigma)$	Distribution: $P(\omega, c)$
1	$e^{j\sigma\mathcal{C}_\omega/2}e^{j\tau\mathcal{W}}e^{j\sigma\mathcal{C}_\omega/2}$	$\int_0^\infty F^*(e^{-\sigma/2}\omega)e^{j\tau\omega}F(e^{\sigma/2}\omega)\,d\omega$	$\frac{1}{2\pi}\int F^*(e^{-\sigma/2}\omega)e^{-j\sigma c}F(e^{\sigma/2}\omega)\,d\sigma$
2	$e^{j\tau\mathcal{W}+j\sigma\mathcal{C}_\omega}$	$\int_0^\infty \exp[2j\tau\omega \sinh(\sigma/2)/\sigma]\,F^*(e^{-\sigma/2}\omega)F(e^{\sigma/2}\omega)\,d\omega$	$\frac{1}{2\pi}\int \frac{\sigma}{2\sinh(\sigma/2)}e^{-j\sigma c}F^*\left(e^{-\sigma/2}\frac{\sigma\omega}{2\sinh(\sigma/2)}\right) \cdot F\left(e^{\sigma/2}\frac{\sigma\omega}{2\sinh(\sigma/2)}\right)d\sigma$
3	$e^{j\tau\mathcal{W}/2}e^{j\sigma\mathcal{C}_\omega}e^{j\tau\mathcal{W}/2}$	$\int_0^\infty \exp(j\tau\omega\cosh(\sigma/2))\,F^*(e^{-\sigma/2}\omega)F(e^{\sigma/2}\omega)\,d\omega$	$\frac{1}{2\pi}\int \frac{1}{\cosh(\sigma/2)}e^{-j\sigma c}F^*\left(e^{-\sigma/2}\frac{\omega}{\cosh(\sigma/2)}\right) \cdot F\left(e^{\sigma/2}\frac{\omega}{\cosh(\sigma/2)}\right)d\sigma$
4	$e^{j\sigma\mathcal{C}_\omega}e^{j\tau\mathcal{W}}$	$\int_0^\infty F^*(e^{-\sigma/2}\omega)\exp(j\tau e^{\sigma/2}\omega)\,F(e^{\sigma/2}\omega)\,d\omega$	$F(\omega)\gamma^*(c,\omega)D_\omega^*(c)$
5	$e^{j\tau\mathcal{W}}e^{j\sigma\mathcal{C}_\omega}$	$\int_0^\infty F^*(e^{-\sigma/2}\omega)\exp(j\tau e^{-\sigma/2}\omega)\,F(e^{\sigma/2}\omega)\,d\omega$	$F^*(\omega)\gamma(c,\omega)D_\omega(c)$
6	$\frac{1}{2}(\{4\}+\{5\})$	$\frac{1}{2}(\{4\}+\{5\})$	Real part $\{F^*(\omega)\gamma(c,\omega)D_\omega(c)\}$

Suppose we chose the characteristic function given by ordering one, then

$$P(t, c) = \frac{1}{4\pi^2}\iiint f^*(e^{\sigma/2}u)\exp(-j\theta t - j\sigma c + j\theta u \cdot \phi(\theta, \sigma))f(e^{-\sigma/2}u)\,d\theta\,du\,d\sigma. \tag{17.3}$$

The particular distribution is obtained, as in the time frequency case, by specifying the kernel. If, for example we take the kernel to be equal to one then we obtain the distribution of Marinovich and Altes. If we take a kernel given by

$$\phi(\theta, \sigma) = \frac{\int_0^\infty f^*(e^{-\sigma/2}t)e^{j\theta t}f(e^{\sigma/2}t)\,dt}{\int_0^\infty \exp[2j\theta t \sinh(\sigma/2)/\sigma]\,f^*(e^{-\sigma/2}t)f(e^{\sigma/2}t)\,dt} \tag{17.4}$$

then one obtains the Bertrand–Bertrand distribution. Note that this kernel is a functional of the signal. That should not be surprising because that is also the case with the time frequency case [6]–[10], although for most of the time frequency cases that have been considered, the kernel is not a functional of the signal.

If we chose the characteristic function given by ordering 2 then

$$P(t, c) = \frac{1}{4\pi^2}\iiint f^*(e^{-\sigma/2}u)\phi(\theta, \sigma) \cdot \exp[-j\theta t - j\sigma c + 2j\theta u \sinh(\sigma/2)/\sigma] \cdot f(e^{\sigma/2}u)\,d\theta\,du\,d\sigma. \tag{17.5}$$

Equation (17.3) or (17.5) may be considered a general class and any other distribution is obtained by appropriate choice of the kernel. So for example, if we chose the kernel equal to one then we obtain the Bertrand–Bertrand distribution while if we chose the kernel to be the inverse of that given by (17.5) then we obtain the Marinovich–Altes distribution.

The identical considerations leading to the time-scale general class will give the general frequency-scale class.

$$P(\omega, c) = \frac{1}{4\pi^2}\iint \phi(\tau, \sigma)M(\tau, \sigma) \cdot e^{-j\tau\omega - j\sigma c}\,d\tau\,d\sigma. \tag{17.6}$$

Choosing M as given by ordering one we have

$$P(\omega, c) = \frac{1}{4\pi^2}\iiint F^*(e^{\sigma/2}u)\exp(-j\tau t + j\sigma c + j\tau u) \cdot \phi(\tau, \sigma)F(e^{-\sigma/2}u)\,d\theta\,du\,d\sigma. \tag{17.7}$$

If we choose M as given by ordering two we obtain

$$P(\omega, c) = \frac{1}{4\pi}\iiint F^*(e^{-\sigma/2}u)\phi(\tau, \sigma) \cdot \exp[-j\tau t + j\sigma c + 2j\tau u \sinh(\sigma/2)/\sigma] \cdot F(e^{\sigma/2}u)\,d\tau\,du\,d\sigma. \tag{17.8}$$

For the time frequency case the conditions on the kernel have been studied and developed in detail [44]–[49].

TABLE IV
TIME-(ω) SCALE REPRESENTATIONS

#	Ordering	Characteristic Function: $M(\theta, \sigma)$	Distribution: $P(t, c)$
1	$e^{j\sigma\mathcal{C}_t/2}e^{j\theta\mathfrak{T}}e^{j\sigma\mathcal{C}_t/2}$	$\int f^*(e^{\sigma/2}t)e^{j\theta t}f(e^{-\sigma/2}t)\,dt$	$\frac{1}{2\pi}\int f^*(e^{-\sigma/2}t)e^{j\sigma c}f(e^{\sigma/2}t)\,d\sigma$
2	$e^{j\theta\mathfrak{T}+j\sigma\mathcal{C}_t}$	$\int f^*(e^{\sigma/2}t)\exp\,[2j\theta t\sinh(\sigma/2)/\sigma]\,f(e^{-\sigma/2}t)\,dt$	$\frac{1}{2\pi}\int \frac{\sigma}{2\sinh(\sigma/2)}e^{j\sigma c}f\left(e^{-\sigma/2}\frac{\sigma t}{2\sinh(\sigma/2)}\right) \cdot f\left(e^{\sigma/2}\frac{\sigma t}{2\sinh(\sigma/2)}\right)d\sigma$
3	$e^{j\theta\mathfrak{T}/2}e^{j\sigma\mathcal{C}_t}e^{j\theta\mathfrak{T}/2}$	$\int f^*(e^{\sigma/2}t)\exp\,(j\theta t\cosh(\sigma/2))\,f(e^{-\sigma/2}t)\,dt$	$\frac{1}{2\pi}\int \frac{1}{\cosh(\sigma/2)}e^{j\sigma c}f^*\left(e^{-\sigma/2}\frac{t}{\cosh(\sigma/2)}\right) \cdot f\left(e^{\sigma/2}\frac{t}{\cosh(\sigma/2)}\right)d\sigma$
4	$e^{j\sigma\mathcal{C}_t}e^{j\theta\mathfrak{T}}$	$\int f^*(e^{\sigma/2}t)\exp\,(-j\theta e^{-\sigma/2}t)\,f(e^{-\sigma/2}t)\,dt$	$\frac{1}{2\pi}f(t)\int e^{-\sigma/2}f^*(e^{-\sigma}t)e^{j\sigma c}\,d\sigma$
5	$e^{j\theta\mathfrak{T}}e^{j\sigma\mathcal{C}_t}$	$\int f^*(e^{\sigma/2}t)\exp\,(j\theta e^{\sigma/2}t)\,f(e^{-\sigma/2}t)\,dt$	$\frac{1}{2\pi}f^*(t)\int e^{\sigma/2}f(e^{\sigma}t)e^{j\sigma c}\,d\sigma$
6	$\frac{1}{2}(\{4\}+\{5\})$	$\frac{1}{2}(\{4\}+\{5\})$	Real part $\{4\}$

Tables 3 and 4 give the distributions of frequency and time for frequency scaling [(ω)scale]. $D(c)$ and D_ω are the time and frequency scale transforms.

Generally speaking, the same ideas apply to other representations. For example, the conditions to satisfy the marginals and instantaneous scale are identical to the conditions for satisfying the marginals and instantaneous frequency in the time-frequency case.

A. Transformation of Distributions

For the time-frequency case the general relation between distributions was first derived in reference [9]. We now do the same for the distributions we have derived for scale. Suppose we have two time-scale distributions P_1 and P_2 with corresponding kernels ϕ_1 and ϕ_2. Their characteristic functions are

$$M_1(\theta, \sigma) = \phi_1(\theta, \sigma)M(\theta, \sigma);$$
$$M_2(\theta, \sigma) = \phi_2(\theta, \sigma)M(\theta, \sigma). \qquad (17.9)$$

Hence,

$$M_1(\theta, \sigma) = \frac{\phi_1(\theta, \sigma)}{\phi_2(\theta, \sigma)}M_2(\theta, \sigma). \qquad (17.10)$$

By taking the Fourier transform of both sides one obtains

$$P_1(t, c) = \iint g(t' - t, c' - c)P_2(t', c')\,dt'\,dc' \qquad (17.11)$$

with

$$g(t, c) = \frac{1}{4\pi^2}\iint e^{j\theta t + j\sigma c}\frac{\phi_1(\theta, \sigma)}{\phi_2(\theta, \sigma)}\,d\theta\,d\sigma. \qquad (17.12)$$

B. Local Autocorrelation Method

A very fruitful approach to study time-frequency distributions has been to generalize the relationship between the energy density spectrum and the deterministic autocorrelation function by defining a local autocorrelation function [1], [10], [44]. We now show that the same can be done for scale. We prove that the general time-scale distributions can be written in the form

$$P(t, c) = \frac{1}{2\pi}\int R_t(\sigma)e^{-jc\sigma}\,d\sigma \qquad (17.13)$$

with

$$R_t(\sigma) = \frac{1}{2\pi}\iint \phi(\theta, \sigma)M(\theta, \sigma)e^{-j\theta t}\,d\theta. \qquad (17.14)$$

We shall call R_t the generalized local scale autocorrelation function. Depending on which definition we use for the general class (17.3)–(17.5) we have respectively

$$R_t(\sigma) = \frac{1}{2\pi}\iint e^{j\theta(u-t)}\phi(\theta, \sigma)f^*(e^{-\sigma/2}u)f(e^{\sigma/2}u)\,du\,d\theta \qquad (17.15)$$

$$R_t(\sigma) = \frac{1}{2\pi}\iint \exp\,[-j\theta t - j\sigma c + 2j\theta u\sinh(\sigma/2)/\sigma] \cdot \phi(\theta, \sigma)f^*(e^{-\sigma/2}u)f(e^{\sigma/2}u)\,du\,d\theta. \qquad (17.16)$$

XVIII. JOINT REPRESENTATIONS OF TIME-FREQUENCY-SCALE

We now consider joint representations of the three variables time, frequency, and scale. There are many orderings possible; we consider only one here and using it we write the general class. Take

$$\mathfrak{M}(\theta, \tau, \sigma) = e^{j\sigma\mathcal{W}/2}e^{j\theta\mathfrak{T} + j\tau\mathcal{W}}e^{j\sigma\mathcal{W}/2}. \qquad (18.1)$$

The characteristic function evaluates to [1]

$$M(\theta, \tau, \sigma) = \int_0^\infty e^{j\theta t}f^*(e^{-\sigma/2}(t - \tfrac{1}{2}\pi)) \cdot f(e^{\sigma/2}(t + \tfrac{1}{2}\tau))\,dt. \qquad (18.2)$$

The distribution is

$$P(t, \omega, c) = \frac{1}{8\pi^2} \iiint M(\theta, \tau, \sigma) \cdot \exp(-j\theta t - j\tau\omega - j\sigma b)\, d\theta\, d\sigma\, d\tau \tag{18.3}$$

which gives

$$P(t, \omega, c) = \frac{1}{4\pi^2} \iint f^*\left(e^{-\sigma/2}\left(t - \frac{1}{2}\tau\right)\right) e^{-j\tau\omega - j\sigma b} \cdot f\left(e^{\sigma/2}\left(t + \frac{1}{2}\tau\right)\right) d\tau\, d\sigma. \tag{18.4}$$

To obtain the general class of time-frequency-scale distributions we multiply the characteristic function by a kernel of three variables, $\phi(\theta, \tau, \sigma)$ to obtain the generalized characteristic function

$$M_{\text{new}}(\theta, \tau, \sigma) = \phi(\theta, \tau, \sigma) M(\theta, \tau, \sigma). \tag{18.5}$$

If we use (18.2) for M we have

$$M_{\text{new}}(\theta, \tau, \sigma) = \phi(\theta, \tau, \sigma) \int e^{j\theta t} f^*(e^{-\sigma/2}(t - \tfrac{1}{2}\tau)) \cdot f(e^{\sigma/2}(t + \tfrac{1}{2}\tau))\, dt. \tag{18.6}$$

and the general class is therefore

$$P(t, \omega, c) = \frac{1}{8\pi^3} \iiint M_{\text{new}}(\theta, \tau, \sigma) \cdot \exp(-j\theta t - j\tau\omega - j\sigma c)\, d\theta\, d\sigma\, d\tau. \tag{18.7}$$

Explicitly,

$$P(t, \omega, c) = \left(\frac{1}{2\pi}\right)^5 \iiiint \phi(\theta, \tau, \sigma) f^* \cdot \left(e^{-\sigma/2}\left(u - \frac{1}{2}\tau\right)\right) \cdot f\left(e^{\sigma/2}\left(u + \frac{1}{2}\tau\right)\right) \exp[-j\theta(t - u) - j\tau\omega - j\sigma c]\, d\theta\, d\sigma\, d\tau\, du. \tag{18.8}$$

This general class of time-frequency-scale distributions satisfies

$$\int P(t, \omega, c)\, dc = P(t, \omega), \tag{18.9}$$

$$\int_0^\infty P(t, \omega, c)\, dt = P(\omega, c), \tag{18.10}$$

$$\int P(t, \omega, c)\, d\omega = P(t, c), \tag{18.11}$$

where the marginals as given by the right hand side of (18.9)–(18.11), are the general classes of time-frequency, frequency-scale and time-scale distributions.

XIX. Frequency-Scaling and Scaling in Other Domains

We have considered the scaling of time and have defined the operator $\mathcal{C}$ to be the time scaling operator. Equally well one can consider the scaling of frequency functions. It is clear from (3.3) that the operator for frequency scaling is, $\mathcal{C}_\omega = -\mathcal{C}$ and its eigenfunctions are $\gamma_\omega(c, \omega)$ which are the same form as the time scaling eigenfunctions with t replaced by ω. For the frequency scaling representation only the positive half of the spectrum is considered, which is equivalent to considering analytic signals. The transformation properties from the frequency domain to the scale domain are now

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int D(c) \frac{e^{jc \ln \omega}}{\sqrt{\omega}}\, d; \qquad \omega \geq 0, \tag{19.1}$$

$$D(c) = \frac{1}{\sqrt{2\pi}} \int_0^\infty F(\omega) \frac{e^{-jc \ln \omega}}{\sqrt{\omega}}\, d\omega. \tag{19.2}$$

Now, everything we have done for the time cases can be directly transliterated to this situation with of course a different interpretation. In particular we can define the short-frequency scale transform by

$$D_\omega(c) = \frac{1}{\sqrt{2\pi}} \int_0^\infty F(\tau) H(\tau - \omega) \frac{e^{-jc \ln \tau}}{\sqrt{\tau}}\, d\tau \tag{19.3}$$

where $H(\omega)$ is the window function in the frequency domain. The joint distribution of frequency and frequency-scale is then $P(t, c) = |D_t(c)|^2$. The same method leading to instantaneous scale now leads to the conditional scale for a given frequency given by $\omega\psi(\omega)$. This differs from scale for a given frequency for the time scaling case by a minus sign. The other quantities such as average scale etc. are obtained the same way except that the limits of integration run from 0 to ∞ in frequency.

Also one can define the density of frequency for a given time by using the short time Fourier transform,

$$E_t(c) = \frac{1}{2\pi} \int_0^\infty \int e^{-j\omega\tau} f(\tau) h(\tau - t) \frac{e^{-jc_\omega \ln \omega}}{\sqrt{\omega}}\, d\tau\, d\omega$$

The joint distribution of time and frequency-scaling is then given by $P(t, c_\omega) = |E_t(c_\omega)|^2$.

A. Joint Frequency Frequency-Scale Representations

In Sections XV–XVII, we obtained joint representations of time and time-scale and frequency and time-scale. We now address how to obtain joint representations of frequency and frequency-scale and time and frequency-scale. Let us first consider the case of frequency and frequency-scaling. The characteristic function will involve the frequency and scale operator. From a mathematical point of view there is no difference to the previous work as long we transliterate appropriately. In fact, all we have to do is to change time to frequency, the signal, $f(t)$ to frequency functions, $F(\omega)$. We have already considered the evaluation of such characteristic functions in Section XVI where we considered distributions to frequency and time scaling. However there the function $F(\omega)$ was the Fourier transform of the signal and in that case it ranged over the whole frequency axis. Now the function, $F(\omega)$ is arbitrary but can only range from zero to infinity. Therefore all we have to do is take the results of Table II and make the limits of integrations zero to infinity. Similarity to obtain distribution of time and frequency scaling we take the results of Table I and let the limits of integration go from $-\infty$ to ∞. These are cataloged in Tables III and IV.

All other consdieration regarding joint representation with frequency scaling can be done the same way. We write here the general classes analogous to (17.3),

$$P(\omega, c_\omega) = \frac{1}{4\pi^2} \iiint F^*(e^{\sigma/2}u) \cdot \exp(-j\tau\omega - j\sigma c_w + j\tau u)\phi(\tau, \sigma) \cdot F(e^{-\sigma/2}u)\, d\tau\, du\, d\sigma \tag{19.4}$$

and also the one analogous to and (17.5).

$$P(\omega, c_w) = \frac{1}{4\pi^2} \iiint F^*(e^{-\sigma/2}u)\phi(\tau, \sigma) \cdot \exp[-j\tau\omega - j\sigma c_w + 2j\tau u \sinh(\sigma/2)/\sigma] \cdot F(e^{\sigma/2}u)\, d\tau\, du\, d\sigma \tag{19.5}$$

In the above two equations we have used c_ω to emphasize that we are doing frequency scaling.

B. Other Domains

In the above, we have shown how to handle frequency scaling. We now want to discuss how to do scaling for an arbitrary domain or physical variable. Suppose the physical variable is a and the functions in that domain are $F(a)$. We define the generalized scaling operator by

$$\mathcal{C}_a = \frac{1}{2j}\left(\frac{d}{da}\, a + a\, \frac{d}{da}\right) \tag{19.6}$$

and in general we have that

$$\begin{aligned} e^{j\sigma\mathcal{C}_a} F(a) &= e^{\sigma/2} F(e^{\sigma} a); \\ e^{j \ln \sigma \mathcal{C}_a} F(a) &= \sqrt{\sigma}\, f(\sigma a). \end{aligned} \tag{19.7}$$

It is clear that nothing we have previously done for time scaling is particular to the time variable. Everything can be carried over directly. To convert the equation we have derived for time scaling to scaling in the "a" domain arbitrary variables a, all we have to do is substitute a for t and $f(a)$ for $f(t)$ in any of formulas we have obtained. The above discussion for the frequency case is an example.

XX. Conclusion

We have presented the basic properties of scale and treated it as a physical variable like frequency and we have given a framework for obtaining joint representations involving scale.

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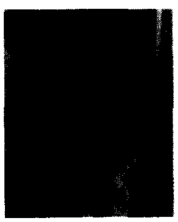

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