THE SPHERICAL INTERPOLATION METHOD FOR CLOSED-FORM PASSIVE SOURCE LOCALIZATION USING RANGE DIFFERENCE MEASUREMENTS

J. S. Abel and J. O. Smith†

Systems Control Technology
1801 Page Mill Road
Palo Alto, California 94303

Abstract. The problem of estimating the *location* of a radiating source from *range difference* measurements taken from a passive, stationary array is discussed. A new closed-form source location estimator, termed the *Spherical Interpolation Estimator*, is presented and analyzed. The location estimates produced by the Spherical Interpolation Estimator are approximate minimizers of a weighted equation error norm, and are shown to approach the maximum likelihood source location estimator.

1. Introduction

The passive localization problem is basic in the fields of underwater acoustics, navigation, aerospace and geophysics. In many situations, measurements of the differences in source-sensor range are available [1,2,3]. These measurements, called range difference (RD) measurements, can be used to infer the location of a source [2,6]. In this paper, we efficiently solve the problem of converting a set of RD measurements to an estimate of source location.

While much literature exists on the problem of estimating RD's from received signals [10], very few papers seem to be available on the problem of converting RD values into source location [3,4,6–9]. In particular, there appear to be no computationally inexpensive source location estimators which have statistically efficient performance for a general source/sensor array geometry [9].

This paper presents a new closed-form location estimator, termed the *Spherical Interpolation (SI) Estimator*. The SI estimator is derived from least-squares techniques, and is a closed-form expression for the minimizer of a weighted "equation error" [11].

2. Problem Formulation

Let N denote the number of sensors, and let d_{ij} denote the RD between sensors i and j $(i,j = 1,\ldots,N)$. The vector of spatial coordinates for the ith sensor is denoted $\underline{x}_i$, and the position of the source is denoted $\underline{x}_s$. We define the coordinate system with $\underline{x}_1$ at the coordinate origin. The *distance* between the source and sensor i is denoted $D_i = ||\underline{x}_i - \underline{x}_s||$, and the distance from the origin to the point $\underline{x}_i$ is denoted R_i. Similarly, $R_s \triangleq ||\underline{x}_s||$. The vector of direction cosines from the origin to the source is denoted by $\underline{\Omega}_s \triangleq \underline{x}_s / R_s$. These quantities appear in Fig. 1. The RD values are given by

$$d_{ij} \triangleq D_i - D_j, \qquad i = 1,\ldots,N, \quad j = 1,\ldots,N \tag{1}$$

The localization problem is to determine $\underline{x}_s$ given d_{ij} for i and j between 1 and N. Note that there are $N(N-1)/2$ distinct RD's d_{ij} (excluding $i = j$, and counting each $d_{ij} \equiv -d_{ji}$ pair once); however, any $N-1$ RD measurements which form a "minimal spanning subtree" determine all the rest (in the noiseless case). The redundancy of the complete set of RD measurements can be used to increase noise immunity. We present here methods for estimating a source location given a set of $N-1$ RD's: d_{i1}, $i = 2,\ldots,N$.

The Maximum Likelihood Solution

Denote by $\underline{\hat{d}}(\underline{\hat{x}}_s)$ the vector of range differences measured from sensor i to sensor 1 predicted by a hypothesized source location $\underline{\hat{x}}_s$ (see Fig. 2), and by $\underline{d}$ the vector of measured RD's d_{i1}. Define $\underline{\epsilon}_{ML}$ as the vector of differences between the measured and hypothesized range differences:

$$\underline{\epsilon}_{ML} = \underline{d} - \underline{\hat{d}}(\underline{\hat{x}}_s) \tag{2}$$

In the absence of *a priori* source location information, it is desirable to estimate the source location as the one which best fits the measurements: e.g., the source location which minimizes some norm of $\underline{\epsilon}_{ML}$. If the measured RD values are corrupted by Gaussian noise with covariance matrix $R_{\underline{d}}$, the maximum likelihood (ML), (minimum-variance, unbiased) estimator of source location is given by [5,8,9]

$$\underline{\hat{x}}_s = \mathrm{Arg}\left[\min_{\underline{x}_s}\left(J_{ML}(\underline{x}_s) \triangleq \underline{\epsilon}_{ML}^T R_{\underline{d}}^{-1} \underline{\epsilon}_{ML}\right)\right] \tag{3}$$

where $\mathrm{Arg}[\min_x f(x)]$ is the value of x which minimizes $f(x)$, and J_{ML} is termed the maximum likelihood cost function. We note that the estimator (3) is the ML source location estimator under the artificial assumptions that 1) only $N-1$ RD's $\underline{d}$ measured relative to an arbitrarily chosen sensor 1 are available, and 2) these RD measurements are Gaussian distributed.

† Dr. Smith is currently with NeXT Inc., 3475 Deer Creek Road, Palo Alto, CA 94304

Geometric Interpretation: Spherical Interpolation

The RD measurement between sensor 1 and sensor i can be interpreted geometrically as the distance from sensor i to the sphere centered at the source, passing through sensor 1, as illustrated in Fig. 3 for the noiseless case. This sphere is a surface of zero RD to sensor 1, for each hypothesized source location $\hat{\underline{x}}_s$. The distance from the sphere to sensor i is therefore $\hat{d}_{i1}$. The ML source location estimate can be seen as $\hat{\underline{x}}_s$ such that $\hat{d}_{i1} \approx d_{i1}$ for every i. In other words, the ML source location estimate is the center of the sphere whose radius and location minimize a quadratic norm of the distance from the surface of the sphere to the points which have a distance d_{i1} away from the ith sensor. The term "Spherical Interpolation" reflects the fact that the optimized spherical surface interpolates through sensor 1. In the absence of niose, the sphere interpolates all RD points.

3. Closed-Form Source Location Estimation

In absence of *a priori* information, the ML estimator, described above, has good statistical properties, providing the minimum-variance, unbiased estimate. However, in general, the ML cost function $J_{ML}(\underline{x}_s)$ is nonconvex in the source location, and to find its minimizer requires computationally expensive global search techniques. For this reason, none of the closed-form solutions discussed in this paper solve (3). Instead, they minimize (or approximately minimize) the L_2 norm of a so-called *equation error* [11], chosen purely to simplify the solution.

3.1. Equation-Error Formulation and Properties

With sensor 1 at the origin, the Pythagorean theorem gives

$$(d_{i1} + R_s)^2 = R_s^2 + R_i^2 - 2\underline{x}_i^T \underline{x}_s \tag{4}$$

(cf. Fig. 1). Moving all terms to the right side of (4), the R_s^2 terms cancel, and we are left with

$$0 = R_i^2 - d_{i1}^2 - 2R_s d_{i1} - 2\underline{x}_i^T \underline{x}_s \tag{5}$$

The first equation is degenerate so we have $N-1$ equations in three unknowns $\underline{x}_s$.

As the delays are not known precisely, an "equation error" [11] is introduced into the left-hand-side of (5)

$$\epsilon_i = R_i^2 - d_{i1}^2 - 2R_s d_{i1} - 2\underline{x}_i^T \underline{x}_s, \quad i = 2, 3, \ldots, N \tag{6}$$

where ϵ_i is the ith component of the equation error. The set of $N-1$ equations (6) can be written in matrix notation as

$$\underline{\epsilon} = \underline{\delta} - 2R_s \underline{d} - 2\mathbf{S}\underline{x}_s \tag{7}$$

where

$$\underline{\delta} \triangleq \begin{pmatrix} R_2^2 - d_{21}^2 \\ R_3^2 - d_{31}^2 \\ \vdots \\ R_N^2 - d_{N1}^2 \end{pmatrix}, \quad \underline{d} \triangleq \begin{pmatrix} d_{21} \\ d_{31} \\ \vdots \\ d_{N1} \end{pmatrix}, \quad \mathbf{S} \triangleq \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_N \end{pmatrix} \tag{8}$$

Relation to $\underline{\epsilon}_{ML}$ and $J_{ML}(\underline{x}_s)$

The equation error vector $\underline{\epsilon}$ is closely related to the ML error vector $\underline{\epsilon}_{ML}$. Adding and subtracting R_s^2 in the definition of the ith equation error (6) gives (upon introducing hats to denote estimated quantities):

$$\begin{aligned} \epsilon_i &= R_i^2 - 2\underline{x}_i^T \hat{\underline{x}}_s + \hat{R}_s^2 - (d_{i1}^2 + 2\hat{R}_s d_{i1} + \hat{R}_s^2) \\ &= (d_{i1} + \hat{d}_{i1} + 2\hat{R}_s)(d_{i1} - \hat{d}_{i1}) \end{aligned} \tag{9}$$

where $\hat{\underline{x}}_s$ and $\hat{R}_s$ are the estimated source location and range, $\underline{x}_i$ is the ith sensor location (known exactly), d_{i1} is the measured range difference, and $\hat{d}_{i1} = ||\underline{x}_i - \hat{\underline{x}}_s|| - \hat{R}_s$ denotes the RD predicted by the source location estimate $\hat{\underline{x}}_s$ (cf. Fig. 2). The term $(d_{i1} - \hat{d}_{i1})$ is that quadratically minimized by the ML source location when the d_{i1} values are measured with Gaussian perturbations. Equation (9) displays the ith equation error as the ith maximum likelihood error $d_{i1} - \hat{d}_{i1}$ times the term $d_{i1} + \hat{d}_{i1} + 2\hat{R}_s$.

From (9), we expect that choosing $\hat{\underline{x}}_s$ as the minimizer of the cost function

$$J_W \triangleq \underline{\epsilon}^T \mathbf{W} \underline{\epsilon} \tag{10}$$

will give an estimate which closely approximates the ML estimate (3) for suitable positive definite $\mathbf{W} \approx \text{diag}[1/(d_{i1} + \hat{d}_{i1} + 2\hat{R}_s)^2] R_{\underline{d}}^{-1}$.

3.2. Spherical Interpolation Methods

Here, projection SI methods and penalty function SI methods for closed-form source location estimation are presented. Projection SI methods find minimizers of the cost function $J_W(\underline{x}_s)$ for a class of weighting matrices $\mathbf{W}$, and penalty function SI methods find approximate minimizers of $J_W(\underline{x}_s)$ for a general weighting matrix.

Projection SI Methods

It is worth noting that the equation equation error (7) is *linear in each of the three variables $\underline{x}_s$, R_s, and $\underline{\Omega}_s \triangleq \underline{x}_s / R_s$ given either of the remaining two.* Therefore, by eliminating $\underline{x}_s$, R_s, or $\underline{\Omega}_s$ from (7), a linear least-squares solution is available for the remaining variables to be estimated. As $\underline{x}_s$, R_s, and $\underline{\Omega}_s$ appear multiplicatively in (7), the appropiate terms can be eliminated from (7) by use of projection operators. Three equivalent techniques are presented below.

Multiplying (7) by projection operators orthogonal to $\underline{d}$, $\underline{\delta}$, and $\mathbf{S}$ give

$$\begin{aligned} \mathbf{P}_{\underline{d}}^{\perp}\underline{\epsilon} &= \mathbf{P}_{\underline{d}}^{\perp}\underline{\delta} - 2\mathbf{P}_{\underline{d}}^{\perp}\mathbf{S}\underline{x}_s \\ -\frac{1}{2R_s}\mathbf{P}_{\underline{\delta}}^{\perp}\underline{\epsilon} &= \mathbf{P}_{\underline{\delta}}^{\perp}\underline{d} + \mathbf{P}_{\underline{\delta}}^{\perp}\mathbf{S}\underline{\Omega}_s \\ \mathbf{P}_{\mathbf{S}}^{\perp}\underline{\epsilon} &= \mathbf{P}_{\mathbf{S}}^{\perp}\underline{\delta} - 2\mathbf{P}_{\mathbf{S}}^{\perp}\underline{d}R_s \end{aligned} \tag{11}$$

where $\mathbf{P}_{\underline{d}}^{\perp}$, $\mathbf{P}_{\underline{\delta}}^{\perp}$, $\mathbf{P}_{\mathbf{S}}^{\perp}$ are idempotent projection operators of rank $N-2$, $N-2$, and $N-4$, respectively, defined by

$$\begin{aligned} \mathbf{P}_{\underline{d}}^{\perp} &\triangleq \mathbf{I} - \underline{d}(\underline{d}^{\mathrm{T}}\underline{d})^{-1}\underline{d}^{\mathrm{T}} \\ \mathbf{P}_{\underline{\delta}}^{\perp} &\triangleq \mathbf{I} - \underline{\delta}(\underline{\delta}^{\mathrm{T}}\underline{\delta})^{-1}\underline{\delta}^{\mathrm{T}} \\ \mathbf{P}_{\mathbf{S}}^{\perp} &\triangleq \mathbf{I} - \mathbf{S}(\mathbf{S}^{\mathrm{T}}\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}} \end{aligned}$$

Note that in (11) the first equation is linear in $\underline{x}_s$, the second is linear in $\underline{\Omega}_s$, and the third is linear in R_s. The source location, range and direction may be estimated using least-squares techniques. Let $\mathbf{V}$ be a positive-definite weighting matrix. Minimizing J_W for $\mathbf{W} = \mathbf{P}_{\underline{d}}^{\perp}\mathbf{V}\mathbf{P}_{\underline{d}}^{\perp}$, $\mathbf{P}_{\underline{\delta}}^{\perp}\mathbf{V}\mathbf{P}_{\underline{\delta}}^{\perp}$, and $\mathbf{P}_{\mathbf{S}}^{\perp}\mathbf{V}\mathbf{P}_{\mathbf{S}}^{\perp}$, we find

$$\begin{aligned} \hat{\underline{x}}_s &= \frac{1}{2}\left(\mathbf{S}^{\mathrm{T}}\mathbf{P}_{\underline{d}}^{\perp}\mathbf{V}\mathbf{P}_{\underline{d}}^{\perp}\mathbf{S}\right)^{-1}\mathbf{S}^{\mathrm{T}}\mathbf{P}_{\underline{d}}^{\perp}\mathbf{V}\mathbf{P}_{\underline{d}}^{\perp}\underline{\delta} \\ \tilde{\underline{\Omega}}_s &= -\left(\mathbf{S}^{\mathrm{T}}\mathbf{P}_{\underline{\delta}}^{\perp}\mathbf{V}\mathbf{P}_{\underline{\delta}}^{\perp}\mathbf{S}\right)^{-1}\mathbf{S}^{\mathrm{T}}\mathbf{P}_{\underline{\delta}}^{\perp}\mathbf{V}\mathbf{P}_{\underline{\delta}}^{\perp}\underline{d} \\ \tilde{R}_s &= \frac{1}{2}\frac{\underline{d}^{\mathrm{T}}\mathbf{P}_{\mathbf{S}}^{\perp}\mathbf{V}\mathbf{P}_{\mathbf{S}}^{\perp}\underline{\delta}}{\underline{d}^{\mathrm{T}}\mathbf{P}_{\mathbf{S}}^{\perp}\mathbf{V}\mathbf{P}_{\mathbf{S}}^{\perp}\underline{d}} \end{aligned} \tag{12}$$

The quantities $\tilde{\underline{\Omega}}_s$ and $\tilde{R}_s$ can be used as estimates of source direction and range, multiplied to estimate $\underline{x}_s$, or used in (7) and (10) to estimate $\underline{x}_s$:

$$\begin{aligned} \tilde{\underline{x}}_{s,1} &= \tilde{R}_s\tilde{\underline{\Omega}}_s \\ \tilde{\underline{x}}_{s,2} &= \frac{1}{2}\left[(\underline{d}+\mathbf{S}\tilde{\underline{\Omega}}_s)^{\mathrm{T}}\mathbf{U}\underline{\delta}/(\underline{d}+\mathbf{S}\tilde{\underline{\Omega}}_s)^{\mathrm{T}}\mathbf{U}(\underline{d}+\mathbf{S}\tilde{\underline{\Omega}}_s)\right]\tilde{\underline{\Omega}}_s \\ \tilde{\underline{x}}_{s,3} &= \frac{1}{2}(\mathbf{S}^{\mathrm{T}}\mathbf{U}\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}\mathbf{U}(\underline{\delta} - 2\tilde{R}_s\underline{d}) \end{aligned} \tag{13}$$

where $\mathbf{U}$ is a positive definite weighting matrix. The estimators defined in (12,13) give source location estimates in closed form as minimizers of the cost function J_W. Since the weighting matrices $\mathbf{W}$ used above are not of full rank, four or more range difference measurements are required to find a unique minimizer of J_W. It appears likely that for the appropriate choices of $\mathbf{V}$ and $\mathbf{U}$, the above estimators can be made identical. Further, note that $\tilde{\underline{\Omega}}_s$ and $\tilde{R}_s$ are designated with tildes, and not hats, since the norm of $\tilde{\underline{\Omega}}_s$ is not necessarily one, and $\tilde{R}_s$ is not necessarily the norm of the resulting $\hat{\underline{x}}_s$. In results not presented here, these discrepancies were seen to be negligible for moderate levels of RD noise.

Penalty Function SI Methods

If $\underline{x}_s$ and R_s are treated as variables to be independently estimated, $\underline{\epsilon}$ can be written as

$$\underline{\epsilon} = \underline{\delta} - 2\begin{bmatrix}\underline{d} & \mathbf{S}\end{bmatrix}\begin{bmatrix}R_s \\ \underline{x}_s\end{bmatrix} \tag{14}$$

and $J_W(\underline{x}_s, R_s)$ (cf. (10)) would be minimized for

$$\begin{bmatrix}\tilde{R}_s \\ \hat{\underline{x}}_s\end{bmatrix} = \frac{1}{2}(\mathbf{\Sigma}^{\mathrm{T}}\mathbf{W}\mathbf{\Sigma})^{-1}\mathbf{\Sigma}^{\mathrm{T}}\mathbf{W}\underline{\delta} \tag{15}$$

where, $\mathbf{\Sigma} \triangleq \begin{bmatrix}\underline{d} & \mathbf{S}\end{bmatrix}$ and $\mathbf{W}$ is a positive definite weighting matrix. This form is a generalization of (12,13).

However, it is desired to incorporate the constraint $R_s = \|\underline{x}_s\|$ into the minimization of J_W. Here, we present a *penalty function method* for estimating $\hat{\underline{x}}_s$ and $\tilde{R}_s$ which allows a cost to be placed on the disparity between the norm of $\hat{\underline{x}}_s$ and $\tilde{R}_s$. Define the penalty function $\mathrm{P}(\underline{x}_s, R_s)$ as

$$\mathrm{P}(\underline{x}_s, R_s) \triangleq \left|\underline{x}_s^{\mathrm{T}}\underline{x}_s - R_s^2\right|$$

Define $\alpha, \overline{\alpha} \geq 0$ such that $\alpha + \overline{\alpha} = 1$. The source location estimate, chosen as the minimizer of $\alpha J_W(\underline{x}_s, R_s) + \overline{\alpha}\mathrm{P}(\underline{x}_s, R_s)$ is seen to be

$$\begin{bmatrix}\tilde{R}_s \\ \hat{\underline{x}}_s\end{bmatrix} = \alpha(2\alpha\mathbf{\Sigma}^{\mathrm{T}}\mathbf{W}\mathbf{\Sigma} \pm \overline{\alpha}\mathbf{M})^{-1}\mathbf{\Sigma}^{\mathrm{T}}\mathbf{W}\underline{\delta} \tag{16}$$

where $\mathbf{M}$ is a diagonal matrix with main diagonal given by $\begin{bmatrix}-1 & 1 & 1 & 1\end{bmatrix}$, and $+$ is taken when $\hat{\underline{x}}_s^{\mathrm{T}}\hat{\underline{x}}_s > \tilde{R}_s^2$. The estimator (16) can be viewed as an approximation to the constrained minimizer of J_W.

Note that the choice of α determines the importance of the discrepency between $\tilde{R}_s$ and $\|\hat{\underline{x}}_s\|$; also when $\alpha = 1$, the solution (15) and the projection solutions can be obtained. Also note, in the case of nonzero $\overline{\alpha}$, the SI estimate (16) requires only three RD measurements to estimate the source location.

4. Conclusion

In this paper, several closed-form expressions for estimating the location of a source given the RD measurement set d_{i1}, from a passive, stationary array, were presented. It was shown that the cost function minimized in obtaining the source location estimates was closely related to the maximum likelihood cost function. In results not presented here, the SI estimators were found to have a mean square error comparable to the Cramer-Rao lower bound [9]. In addition, other results showed the SI estimators to have smaller bias and variance than two other closed-form methods [8]. As a final note, when a complete set of RD measurements d_{ij}, $i,j = 1,\ldots,N$ is available, the source location may be estimated by averaging the N SI source location estimates made using each sensor as a reference sensor.

5. References

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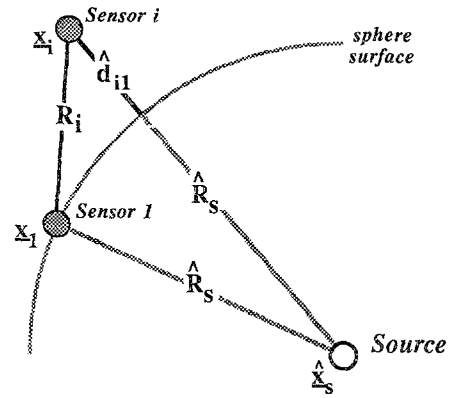

Figure 2
Range difference mesurements predicted by a hypothesized source location

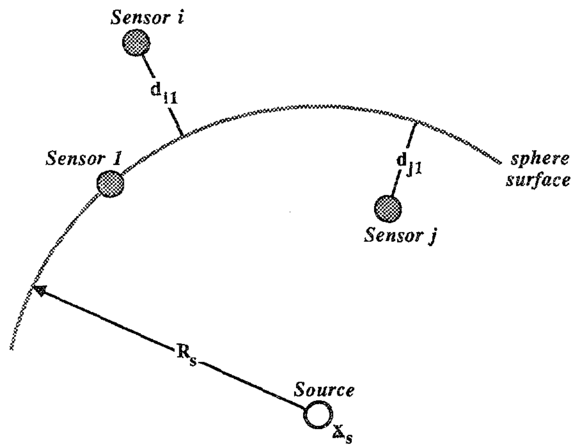

Figure 3
Spherical Interpolation interpretation of range difference measurements

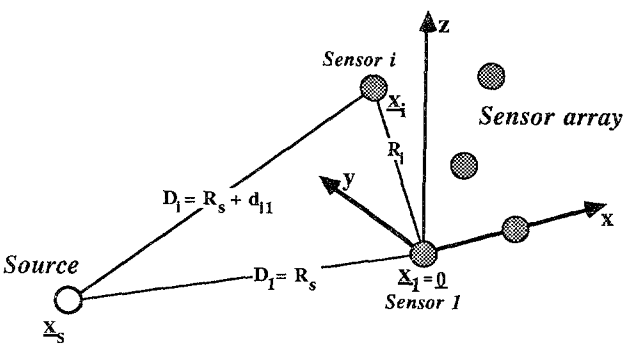

Figure 1
Source/sensor array geometry