
ELEN E4810: Digital Signal Processing

Topic 9:

Filter Design: FIR

1. Windowed Impulse Response
2. Window Shapes
3. Design by Iterative Optimization



1. FIR Filter Design

- FIR filters
 - no poles (just zeros)
 - no precedent in analog filter design
- Approaches
 - windowing ideal impulse response
 - iterative (computer-aided) design



Least Integral-Squared Error

- Given desired FR $H_d(e^{j\omega})$, what is the **best** finite $h_t[n]$ to approximate it?

best in what sense?

- Can try to minimize **Integral Squared Error** (ISE) of frequency responses:

$$\phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_d(e^{j\omega}) - H_t(e^{j\omega})|^2 d\omega$$

$= \text{DTFT}\{h_t[n]\}$



Least Integral-Squared Error

- Ideal IR is $h_d[n] = \text{IDTFT}\{H_d(e^{j\omega})\}$,
(usually infinite-extent)
- By **Parseval**, **ISE** $\phi = \sum_{n=-\infty}^{\infty} |h_d[n] - h_t[n]|^2$
- But: $h_t[n]$ only exists for $n = -M..M$,

$$\Rightarrow \phi = \sum_{n=-M}^M |h_d[n] - h_t[n]|^2 + \sum_{n < -M, n > M} |h_d[n]|^2$$

minimized by making
 $h_t[n] = h_d[n], -M \leq n \leq M$

not altered by $h_t[n]$

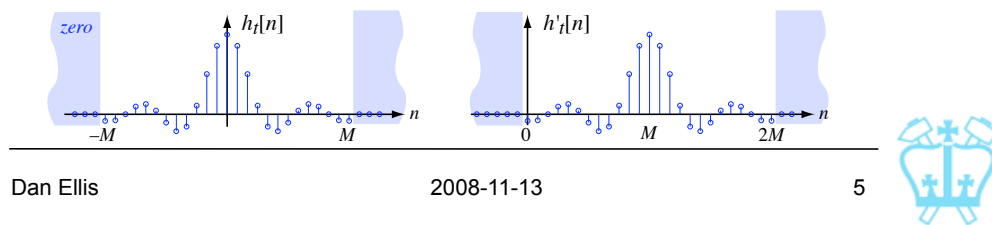


Least Integral-Squared Error

- Thus, minimum mean-squared error approximation in $2M+1$ point FIR is truncated IDTFT:

$$h_t[n] = \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

- Make **causal** by delaying by M points
 $\rightarrow h'_t[n] = 0$ for $n < 0$



Dan Ellis

2008-11-13

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Approximating Ideal Filters

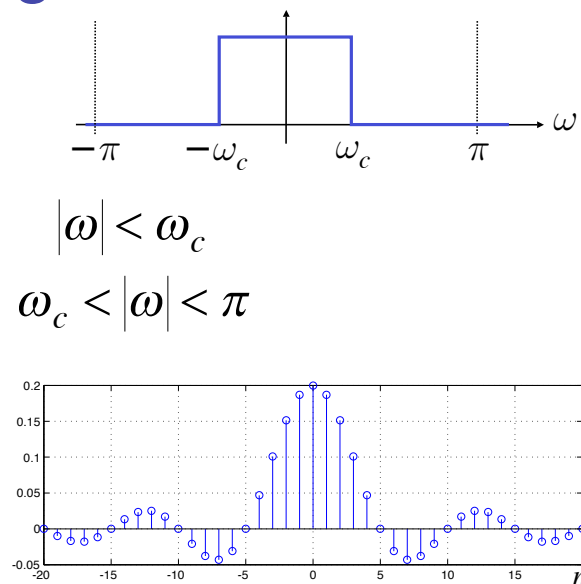
- From topic 6, **ideal lowpass** has:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

and:

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}$$

(doubly infinite)



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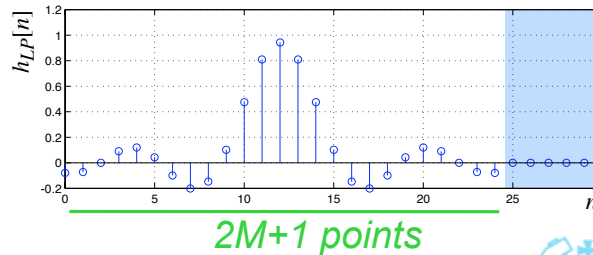
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Approximating Ideal Filters

- Thus, **minimum ISE causal** approximation to an **ideal lowpass**

$$\hat{h}_{LP}[n] = \begin{cases} \frac{\sin \omega_c (n - M)}{\pi(n - M)} & 0 \leq n \leq 2M \\ 0 & \text{otherwise} \end{cases}$$

Causal shift

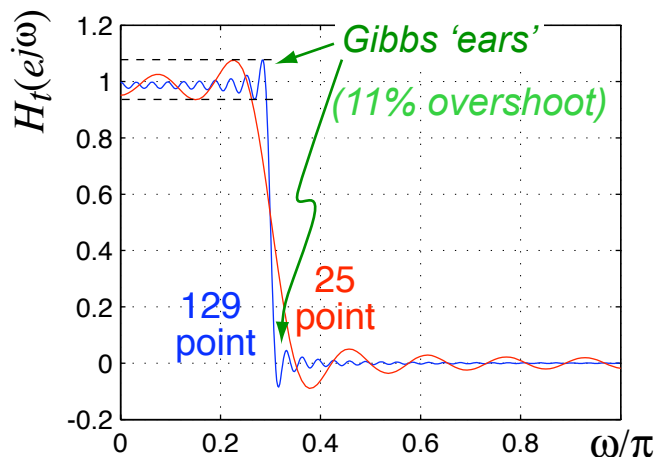


Gibbs Phenomenon

- Truncated ideal filters have **Gibbs' Ears**:

Increasing filter length
 → narrower ears
 (reduces ISE)
but height the same

→ **not optimal by**
minimax criterion

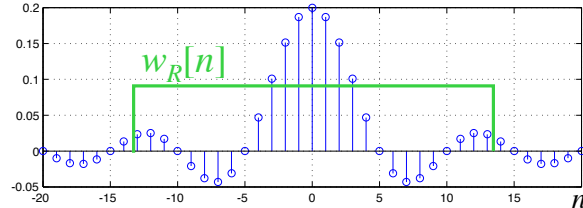


Where Gibbs comes from

- Truncation of $h_d[n]$ to $2M+1$ points is **multiplication** by a **rectangular window**:

$$h_t[n] = h_d[n] \cdot w_R[n]$$

$$w_R[n] = \begin{cases} 1 & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$



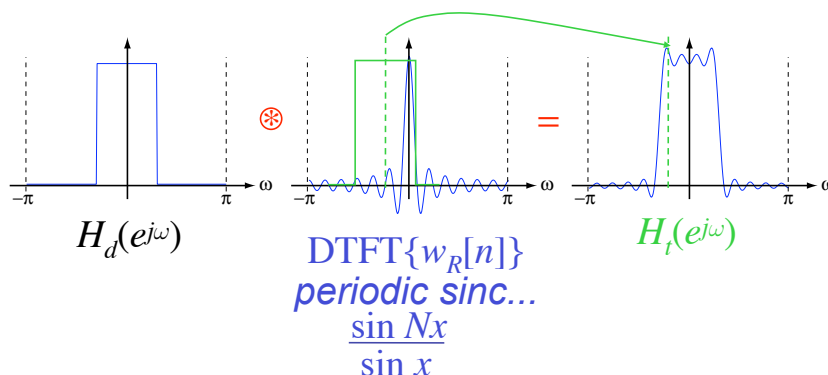
- Multiplication** in time domain is **convolution** in frequency domain:

$$g[n] \cdot h[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega-\theta)}) d\theta$$



Where Gibbs comes from

- Thus, FR of **truncated** response is **convolution** of ideal FR and **FR of rectangular window** (pdc.sinc):



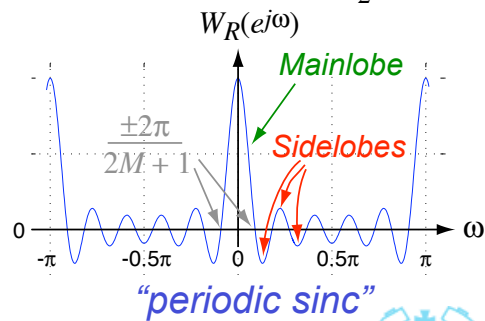
Where Gibbs comes from

- Rectangular window:

$$w_R[n] = \begin{cases} 1 & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases} \Rightarrow W_R(e^{j\omega}) = \sum_{n=-M}^M e^{-jn\omega} = \frac{\sin\left([2M+1]\frac{\omega}{2}\right)}{\sin\frac{\omega}{2}}$$

- Mainlobe width ($\propto 1/L$) determines transition band
- Sidelobe height determines ripples

doesn't vary with length



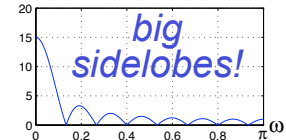
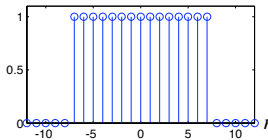
2. Window Shapes for Filters

- Windowing (infinite) ideal response
→ FIR filter: $h_t[n] = h_d[n] \cdot w[n]$
- Rectangular window has best ISE error
- Other "tapered windows" vary in:
 - mainlobe → transition band width
 - sidelobes → size of ripples near transition
- Variety of 'classic' windows...

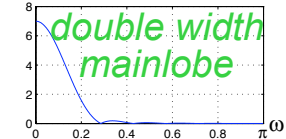
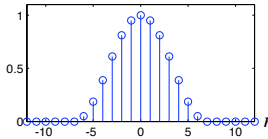


Window Shapes for FIR Filters

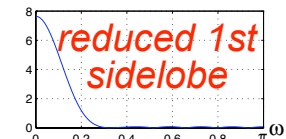
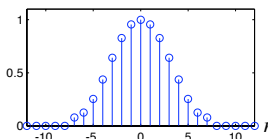
■ **Rectangular:**
 $w[n] = 1 \quad -M \leq n \leq M$



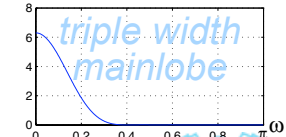
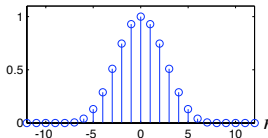
■ **Hann:**
 $0.5 + 0.5 \cos(2\pi \frac{n}{2M+1})$



■ **Hamming:**
 $0.54 + 0.46 \cos(2\pi \frac{n}{2M+1})$

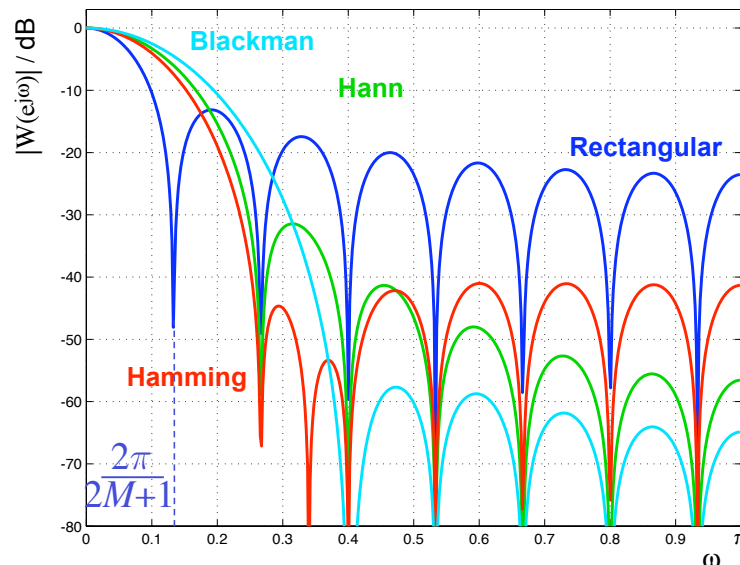


■ **Blackman:**
 $0.42 + 0.5 \cos(2\pi \frac{n}{2M+1})$
 $+ 0.08 \cos(2\pi \frac{2n}{2M+1})$



Window Shapes for FIR Filters

■ **Comparison on dB scale:**



Adjustable Windows

- So far, **discrete** main-sidelobe tradeoffs..
- Kaiser window** = parametric, **continuous** tradeoff:

$$w[n] = \frac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\beta)} \quad -M \leq n \leq M$$

modified zero-order Bessel function

- Empirically, for min. SB atten. of α dB:

$$\beta = \begin{cases} 0.11(\alpha - 8.7) & 50 < \alpha \\ 0.58(\alpha - 21)^{0.4} & 21 \leq \alpha \leq 50 \\ +0.08(\alpha - 21) & \alpha < 21 \\ 0 & \alpha < 21 \end{cases}$$

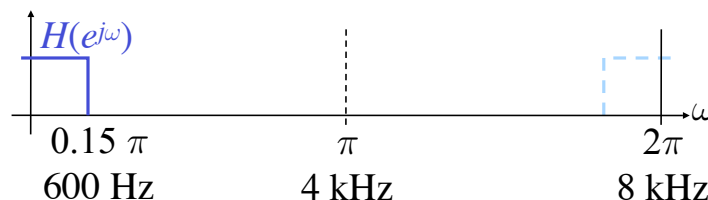
required order $N = \frac{\alpha - 8}{2.3\Delta\omega}$

transition width



Windowed Filter Example

- Design a 25 point FIR low-pass filter with a cutoff of 600 Hz (SR = 8 kHz)
- No specific transition/ripple req's
→ compromise: use **Hamming** window
- Convert the frequency to radians/sample: $\omega_c = \frac{600}{8000} \times 2\pi = 0.15\pi$



Windowed Filter Example

1. Get ideal filter impulse response:

$$\omega_c = 0.15\pi \Rightarrow h_d[n] = \frac{\sin 0.15\pi n}{\pi n}$$

2. Get window:

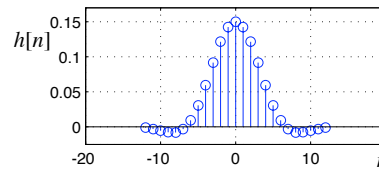
Hamming @ $N = 25 \rightarrow M = 12$ ($N = 2M + 1$)

$$\Rightarrow w[n] = 0.54 + 0.46 \cos\left(2\pi \frac{n}{25}\right) \quad -12 \leq n \leq 12$$

3. Apply window:

$$h[n] = h_d[n] \cdot w[n]$$

$$= \frac{\sin 0.15\pi n}{\pi n} \left(0.54 + 0.46 \cos \frac{2\pi n}{25}\right) \quad -12 \leq n \leq 12$$



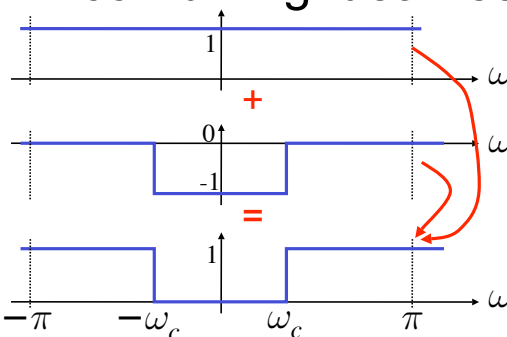
Freq. Resp. (FR) Arithmetic

- Ideal LPF has **pure-real** FR i.e.

$$\theta(\omega) = 0, H(e^{j\omega}) = |H(e^{j\omega})|$$

→ Can build **piecewise-constant** FRs by combining ideal responses, e.g. HPF:

wouldn't work if phases were nonzero!



$$\delta[n] \quad \text{i.e. } H(e^{j\omega}) = 1$$

+

$$-h_{LP}[n] \quad H_{LP}(e^{j\omega}) = 1 \text{ for } |\omega| < \omega_c$$

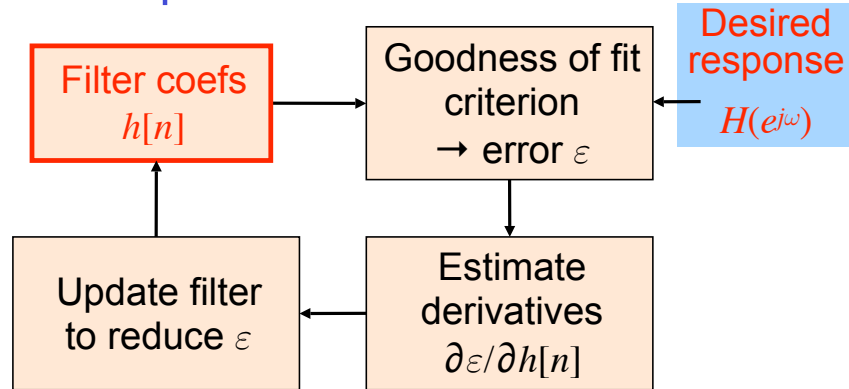
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$$h_{HP}[n] = \delta[n] - (\sin \omega_c n) / \pi n$$



3. Iterative FIR Filter Design

- Can derive filter coefficients by iterative optimization:



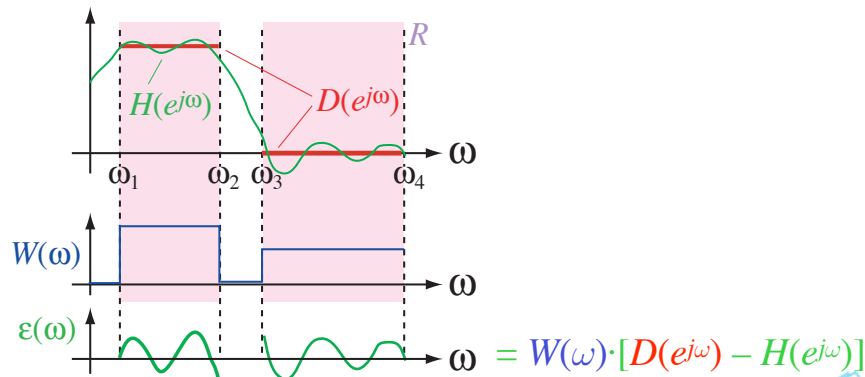
- Gradient descent / nonlinear optimiz'n



Error Criteria

$$\epsilon = \int_{\omega \in R} |W(\omega) \cdot [D(e^{j\omega}) - H(e^{j\omega})]|^p d\omega$$

error measurement region \rightarrow $\omega \in R$
error weighting \rightarrow $W(\omega)$
desired response \rightarrow $D(e^{j\omega})$
actual response \rightarrow $H(e^{j\omega})$
exponent:
 $2 \rightarrow$ least sq
 $\infty \rightarrow$ minimax



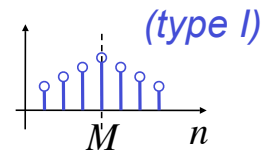
Minimax FIR Filters

- Iterative design of FIR filters with:

- equiripple (minimax criterion)

- linear-phase

→ symmetric IR $h[n] = (-)h[-n]$



- Recall: symmetric FIR filters have FR

$$H(e^{j\omega}) = e^{-j\omega M} \tilde{H}(\omega) \text{ with pure-real}$$

$$\tilde{H}(\omega) = \sum_{k=0}^M a[k] \cos(k\omega) \quad \begin{array}{l} a[0] = h[M] \\ a[k] = 2h[M-k] \end{array}$$

i.e. combo of cosines of **multiples of ω**



Minimax FIR Filters

- Now, $\cos(k\omega)$ can be expressed as a polynomial in $\cos(\omega)^k$ and lower powers

- e.g. $\cos(2\omega) = 2(\cos\omega)^2 - 1$

- Thus, we can find α 's such that

$$\tilde{H}(\omega) = \sum_{k=0}^M a[k] \cos(k\omega) = \sum_{k=0}^M \alpha[k] (\cos\omega)^k$$

Mth order polynomial in $\cos\omega$

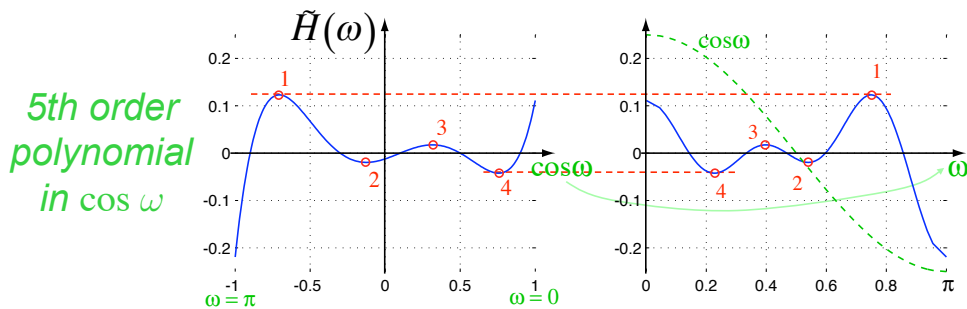
- $\alpha[k]$ s are simply related to $a[k]$ s



Minimax FIR Filters

$$\tilde{H}(\omega) = \sum_{k=0}^M \alpha[k](\cos \omega)^k \quad \text{\textit{M}^{th} order polynomial in } \cos \omega$$

- An M^{th} order polynomial has at most $M - 1$ maxima and minima:



$\Rightarrow \tilde{H}(\omega)$ has at most $M-1$ min/max (ripples)



Alternation Theorem

- Key ingredient to Parks-McClellan:

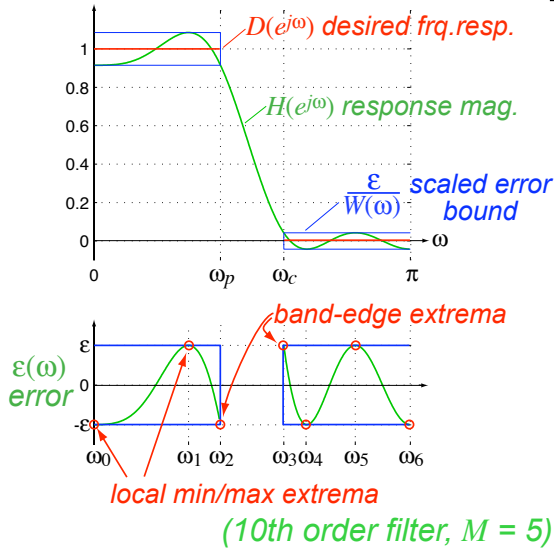
$\tilde{H}(\omega)$ is the **unique, best**, weighted-minimax order $2M$ approx. to $D(e^{j\omega})$

- \Leftrightarrow
- $\tilde{H}(\omega)$ has at least $M+2$ “**extremal**” freqs $\omega_0 < \omega_1 < \dots < \omega_M < \omega_{M+1}$ over ω subset R
 - error magnitude is **equal** at each extremal: $|\varepsilon(\omega_i)| = \varepsilon \quad \forall i$
 - peak error **alternates** in sign: $\varepsilon(\omega_i) = -\varepsilon(\omega_{i+1})$



Alternation Theorem

- Hence, for a frequency response:



- If** $\epsilon(\omega)$ reaches a **peak error** magnitude ϵ at some set of **extremal frequencies** ω_i
- And** the **sign** of the peak error **alternates**
- And** we have at least $M + 2$ of them
- Then** **optimal minimax**



Alternation Theorem

- By Alternation Theorem, $M+2$ **extrema** of **alternating** signs \Rightarrow optimal minimax filter
- But** $\tilde{H}(\omega)$ has at most $M-1$ extrema \Rightarrow need at least **3** more from **band edges**
- 2 bands give **4** band edges \Rightarrow can afford to “miss” only **one**
- Alternation** rules out **transition band edges**, thus have 1 or 2 **outer edges**



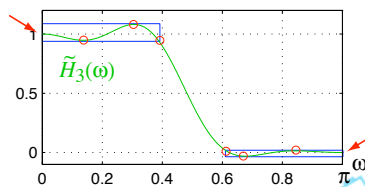
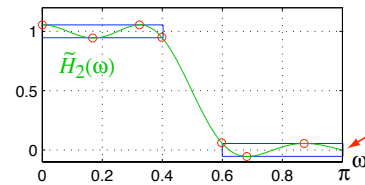
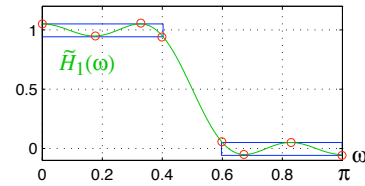
Alternation Theorem

- For $M = 5$ (10th order):

- 8 extrema ($M+3$, 4 band edges) - **great!**

- 7 extrema ($M+2$, 3 band edges) - **OK!**

- 6 extrema ($M+1$, only 2 transition band edges) → **NOT OPTIMAL**



Parks-McClellan Algorithm

- To recap:

- FIR CAD constraints

$$D(e^{j\omega}), W(\omega) \rightarrow \varepsilon(\omega)$$

- Zero-phase FIR

$$\tilde{H}(\omega) = \sum_k \alpha_k \cos^k \omega \rightarrow M-1 \text{ min/max}$$

- Alternation theorem

optimal → $\geq M+2$ pk errs, alter'ng sign

- Hence, can **spot** 'best' filter when we see it – but how to **find** it?



Parks-McClellan Algorithm

- **Alternation** → $[\tilde{H}(\omega) - \tilde{D}(\omega)]/W(\omega)$ must = $\pm\epsilon$ at $M+2$ (unknown) frequencies $\{\omega_i\}$...
- Iteratively update $h[n]$ with **Remez exchange algorithm**:
 - estimate/guess $M+2$ extremals $\{\omega_i\}$
 - solve for $\alpha[n], \epsilon$ (→ $h[n]$)
 - find actual min/max in $\epsilon(\omega)$ → new $\{\omega_i\}$
 - repeat until $|\epsilon(\omega_i)|$ is constant
- Converges rapidly!



Parks-McClellan Algorithm

- In Matlab,

```
>> h=remez(10, [0 0.4 0.6 1],
           [1 1 0 0],
           [1 2])
```

filter order (2M) points to 10
band edges $\div \pi$ points to [0 0.4 0.6 1]
desired magnitude at band edges points to [1 1 0 0]
error weights per band points to [1 2]

