
ELEN E4810: Digital Signal Processing

Topic 6:

Filters - Introduction

1. Simple Filters
2. Ideal Filters
3. Linear Phase and FIR filter types



1. Simple Filters

- **Filter** = system for altering signal in some 'useful' way
- **LSI** systems:
 - are characterized by $H(z)$ (or $h[n]$)
 - have different **gains** (& **phase shifts**) at different **frequencies**
 - can be **designed** systematically for specific filtering tasks



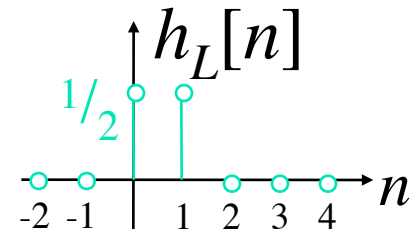
FIR & IIR

- FIR = **finite impulse response**
 - ⇔ **no feedback** in block diagram
 - ⇔ **no poles** (only zeros)
- IIR = **infinite impulse response**
 - ⇔ **feedback** in block diagram
 - ⇔ **poles** (and often zeros)

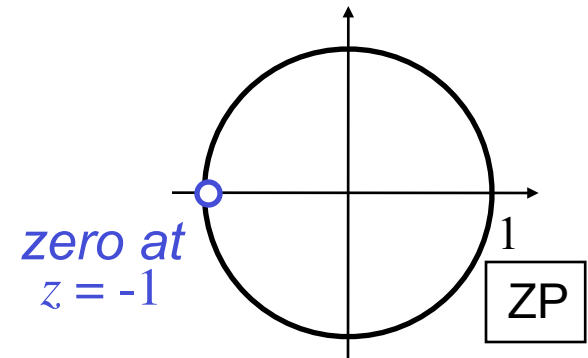


Simple FIR Lowpass

- $h_L[n] = \left\{ \underset{\uparrow}{1/2} \quad 1/2 \right\}$
(2 pt moving avg.)



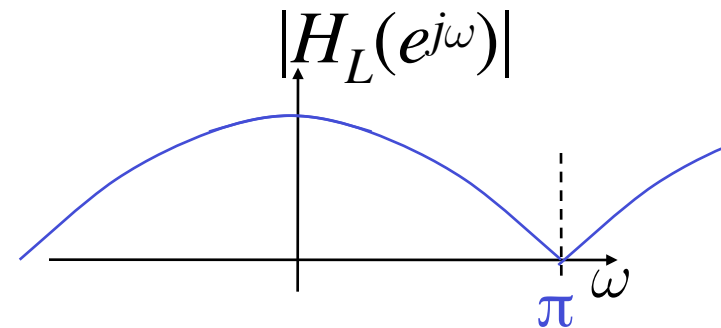
$$H_L(z) = \frac{1}{2} (1 + z^{-1}) = \frac{z + 1}{2z}$$



$$\Rightarrow H_L(e^{j\omega}) = e^{-j\omega/2} \cos(\omega/2)$$

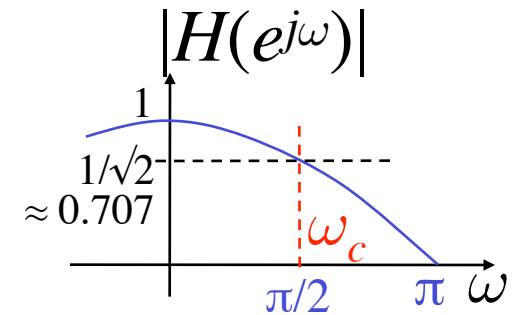
1/2 sample delay (pointing to $e^{-j\omega/2}$)

$e^{j\omega/2} + e^{-j\omega/2}$ (pointing to $\cos(\omega/2)$)



Simple FIR Lowpass

- Filters are often characterized by their **cutoff frequency** ω_c :



- Cutoff frequency is most often defined as the **half-power point** i.e.

$$\left|H(e^{j\omega_c})\right|^2 = \frac{1}{2} \max \left\{ \left|H(e^{j\omega})\right|^2 \right\} \Rightarrow H = \frac{1}{\sqrt{2}} H_{\max}$$

- If $\left|H(e^{j\omega})\right| = \cos(\omega/2)$

$$\text{then } \omega_c = 2 \cos^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{2}$$



deciBels

- Filter magnitude responses are often described in deciBels (dB)
- dB is simply a scaled log value:

$$dB = 20 \log_{10}(\textit{level}) = 10 \log_{10}(\textit{power}) \quad \textit{power} = \textit{level}^2$$

- Half-power also known as **3dB point**:

$$|H|_{\textit{cutoff}} = \frac{1}{\sqrt{2}} |H|_{\textit{max}}$$

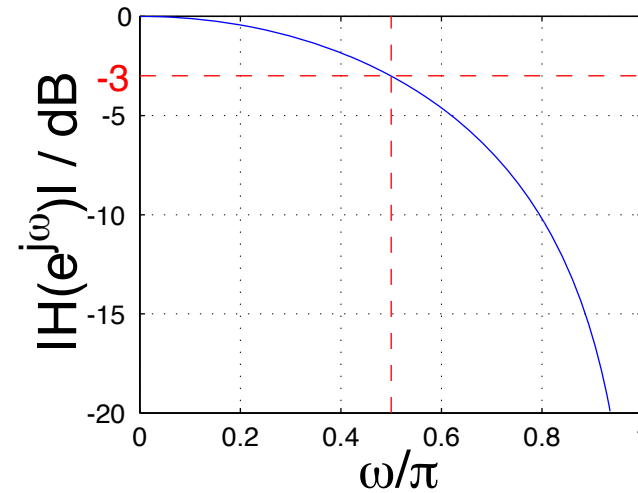
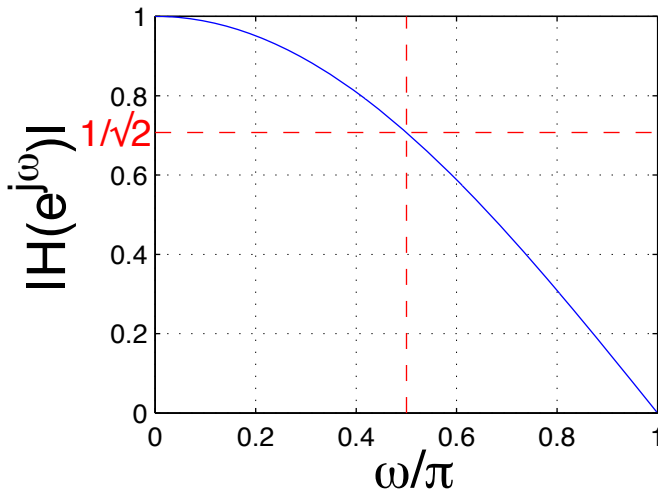
$$dB\{|H|_{\textit{cutoff}}\} = dB\{|H|_{\textit{max}}\} + 20 \log_{10}\left(\frac{1}{\sqrt{2}}\right)$$

$$= dB\{|H|_{\textit{max}}\} - 3.01$$



deciBels

- We usually plot magnitudes in dB:



- A gain of **0** corresponds to **$-\infty$ dB**



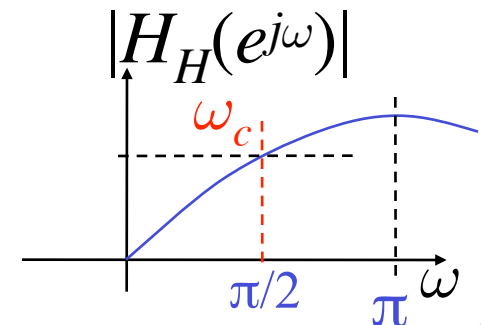
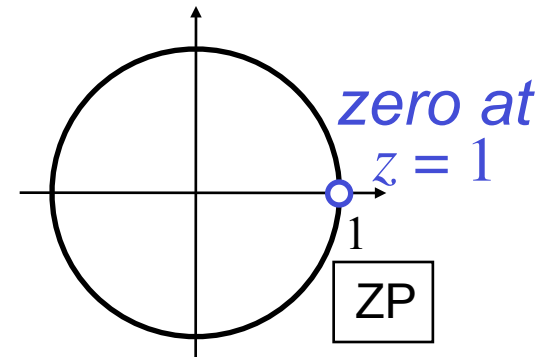
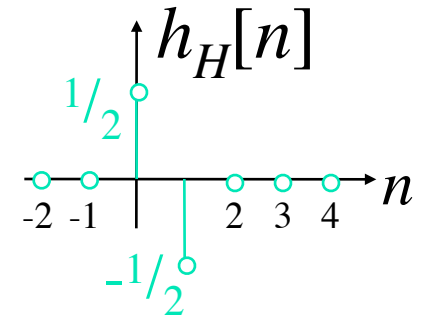
Simple FIR Highpass

- $h_H[n] = \{1/2 \ -1/2\}$

$$H_H(z) = \frac{1}{2}(1 - z^{-1}) = \frac{z-1}{2z}$$

$$\Rightarrow H_H(e^{j\omega}) = je^{-j\omega/2} \sin(\omega/2)$$

- 3dB point $\omega_c = \pi/2$ (again)



FIR Lowpass and Highpass

- Note:

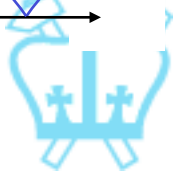
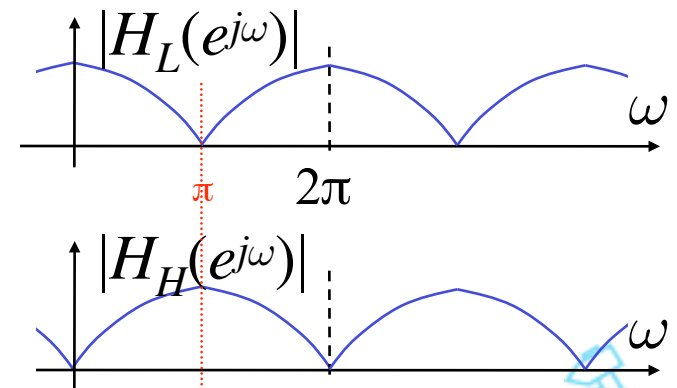
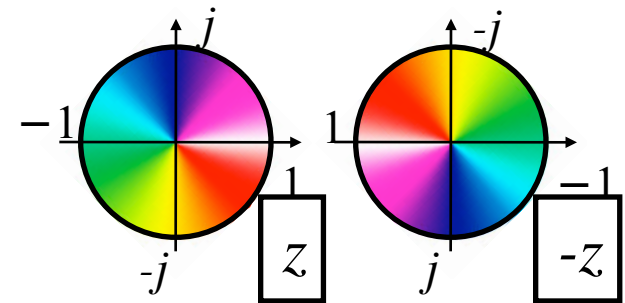
$$h_L[n] = \{1/2 \ 1/2\} \quad h_H[n] = \{1/2 \ -1/2\}$$

- i.e. $h_H[n] = (-1)^n h_L[n]$

$$\Rightarrow H_H(z) = H_L(-z)$$

- i.e. **180° rotation** of the z-plane,

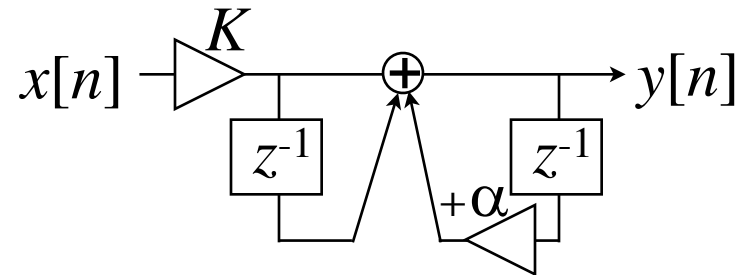
\Rightarrow **π shift** of frequency response



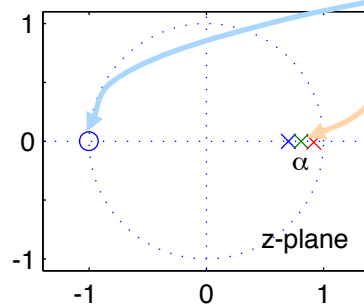
Simple IIR Lowpass

IIR → feedback, zeros **and poles**,
conditional stability, $h[n]$ less useful

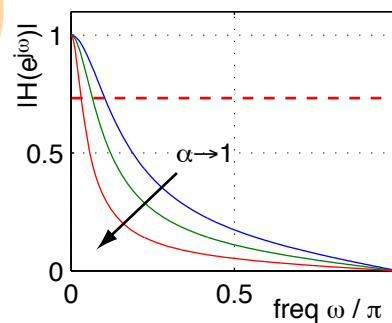
$$H_{LIP}(z) = K \frac{1 + z^{-1}}{1 - \alpha z^{-1}}$$



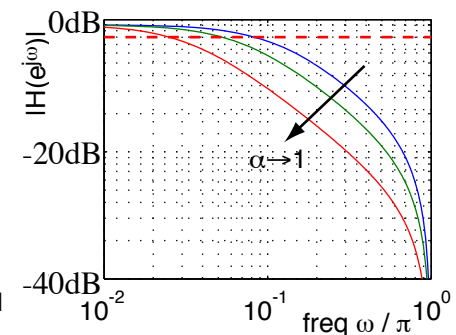
scale to make
gain = 1 at $\omega = 0$
→ $K = (1 - \alpha)/2$



*pole-zero
diagram*



*frequency
response*



*FR on
log-log axes*



Simple IIR Lowpass

$$H_{LP}(z) = K \frac{1 + z^{-1}}{1 - \alpha z^{-1}}$$

max = 1
using $K = (1 - \alpha)/2$

- **Cutoff freq. ω_c** from $|H_{LP}(e^{j\omega_c})|^2 = \frac{\max}{2}$

$$\Rightarrow \frac{(1 - \alpha)^2}{4} \frac{(1 + e^{-j\omega_c})(1 + e^{j\omega_c})}{(1 - \alpha e^{-j\omega_c})(1 - \alpha e^{j\omega_c})} = \frac{1}{2}$$

$$\Rightarrow \cos \omega_c = \frac{2\alpha}{1 + \alpha^2} \Rightarrow \alpha = \frac{1 - \sin \omega_c}{\cos \omega_c}$$

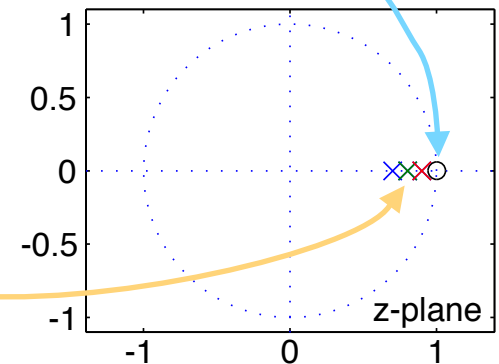
Design Equation



Simple IIR Highpass

$$H_{HP}(z) = K \frac{1 - z^{-1}}{1 - \alpha z^{-1}}$$

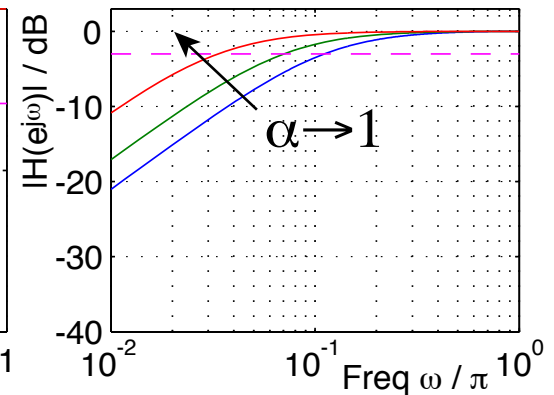
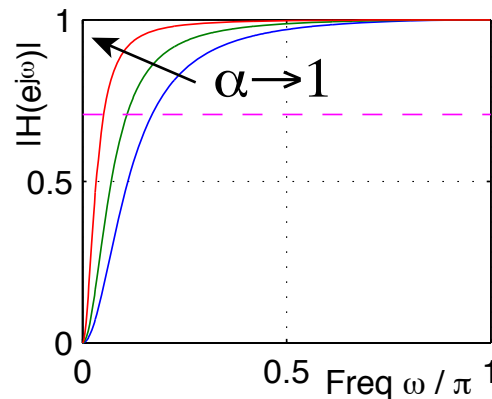
Pass $\omega = \pi \rightarrow H_{HP}(-1) = 1$
 $\rightarrow K = (1 + \alpha)/2$



Design Equation:

$$\alpha = \frac{1 - \sin \omega_c}{\cos \omega_c}$$

(again)



Highpass and Lowpass

- Consider lowpass filter:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & \omega \approx 0 \\ \sim 0 & \text{large } \omega \end{cases}$$

- Then:

$$\underbrace{1 - H_{LP}(e^{j\omega})}_{\text{just another } z \text{ poly}} = \begin{cases} 0 & \omega \approx 0 \\ \sim 1 & \text{large } \omega \end{cases} \quad \begin{array}{l} \bullet \text{ Highpass} \\ \bullet c/w (-1)^n h[n] \end{array}$$

- However, $|1 - H_{LP}(z)| \neq 1 - |H_{LP}(z)|$
(unless $H(e^{j\omega})$ is pure real - not for IIR)



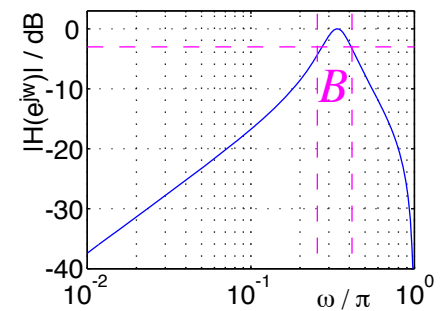
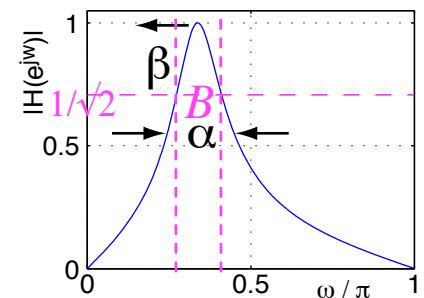
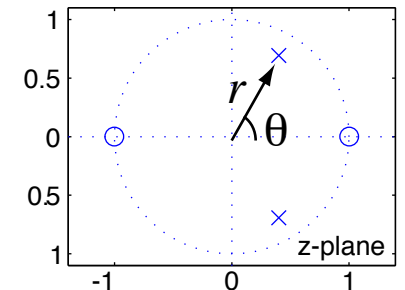
Simple IIR Bandpass

$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

$$= K \frac{(1+z^{-1})(1-z^{-1})}{1-2r\cos\theta \cdot z^{-1}+r^2 z^{-2}}$$

where $r = \sqrt{\alpha}$ $\cos\theta = \frac{\beta(1+\alpha)}{2\sqrt{\alpha}}$

Design $\left(\begin{array}{l} \text{Center freq } \omega_c = \cos^{-1} \beta \\ \text{3dB bandwidth } B = \cos^{-1} \left(\frac{2\alpha}{1+\alpha^2} \right) \end{array} \right.$



Simple Filter Example

- Design a second-order IIR bandpass filter with $\omega_c = 0.4\pi$, 3dB b/w of 0.1π

$$\omega_c = 0.4\pi \Rightarrow \beta = \cos \omega_c = 0.3090$$

$$B = 0.1\pi \Rightarrow \frac{2\alpha}{1 + \alpha^2} = \cos(0.1\pi) \Rightarrow \alpha = 0.7265$$

$$\Rightarrow H_{BP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}}$$

$$= \frac{0.1367(1 - z^{-2})}{1 - 0.5335z^{-1} + 0.7265z^{-2}}$$

sensitive..



Simple IIR Bandstop

zeros at ω_c (per $1 - 2r \cos\theta z^{-1} + r^2 z^{-2}$)

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}}$$

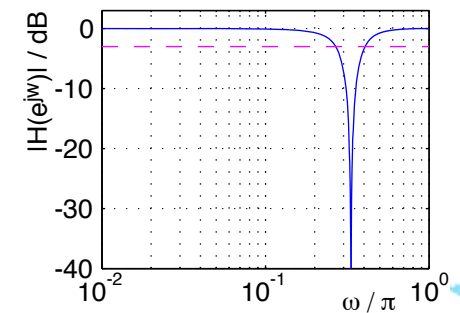
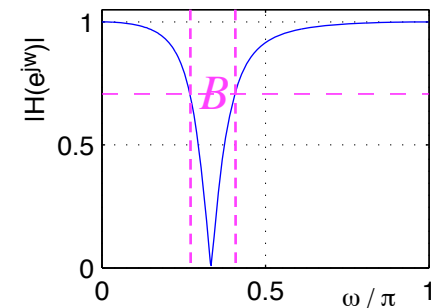
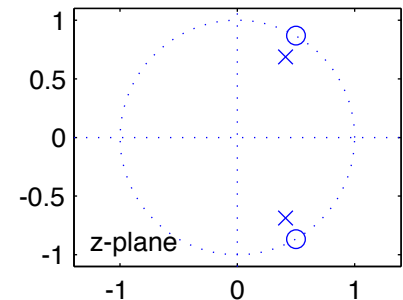
same poles as H_{BP}

- Design eqns:

$$\omega_c = \cos^{-1} \beta \Rightarrow \beta = \cos \omega_c$$

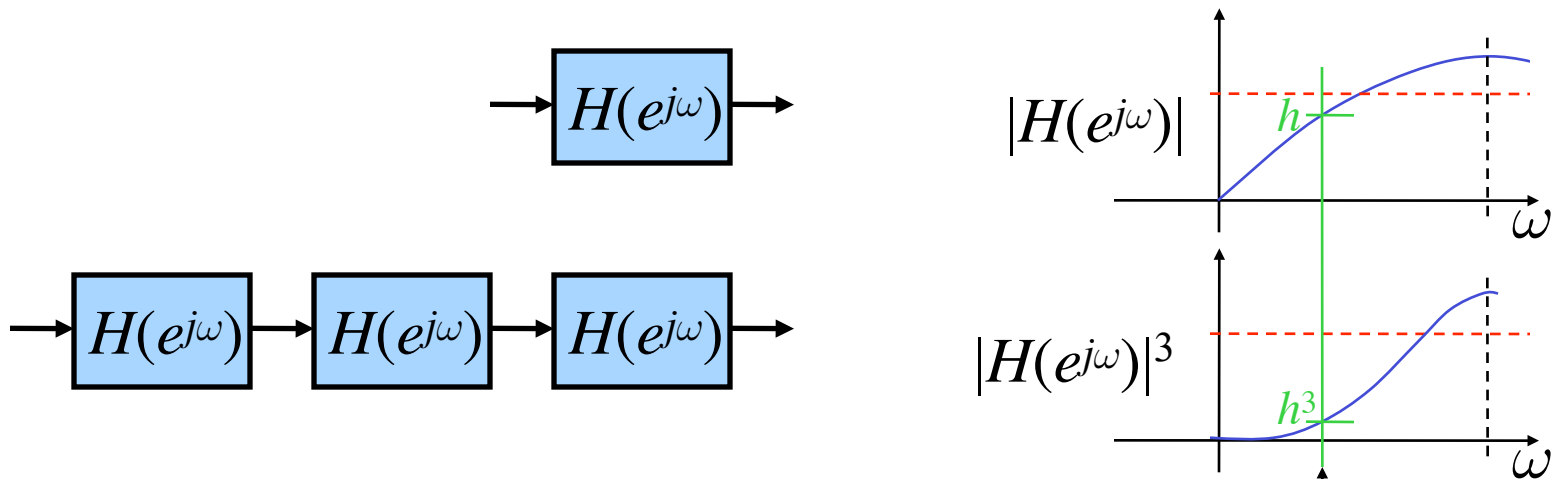
$$B = \cos^{-1} \left(\frac{2\alpha}{1 + \alpha^2} \right)$$

$$\Rightarrow \alpha = \frac{1}{\cos B} - \sqrt{\frac{1}{\cos^2 B} - 1}$$

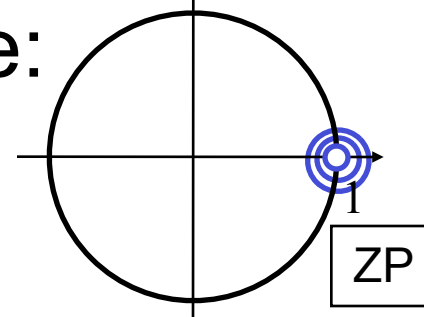


Cascading Filters

- Repeating a filter (**cascade** connection) makes its characteristics more abrupt:

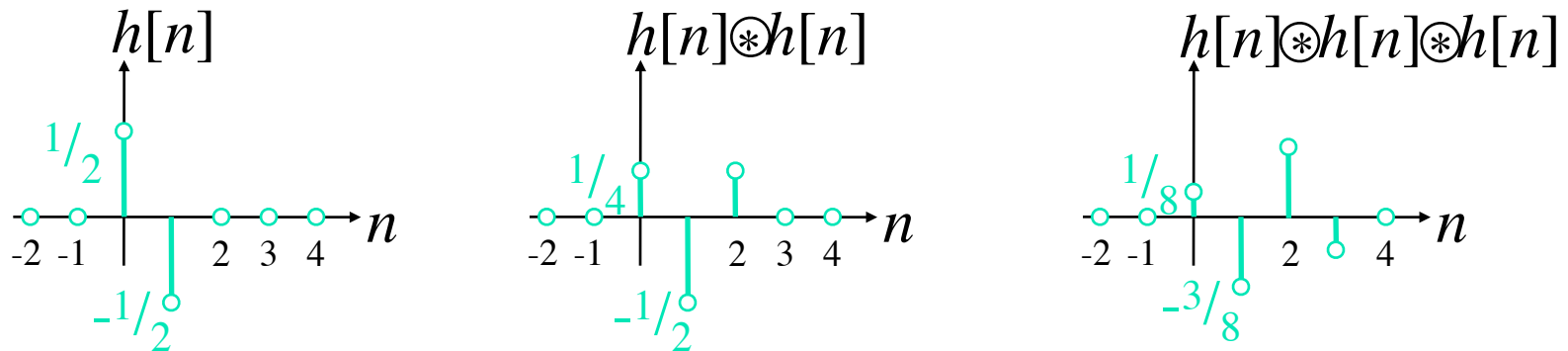


- Repeated roots in z-plane:



Cascading Filters

- Cascade systems are **higher order**
e.g. longer (finite) impulse response:

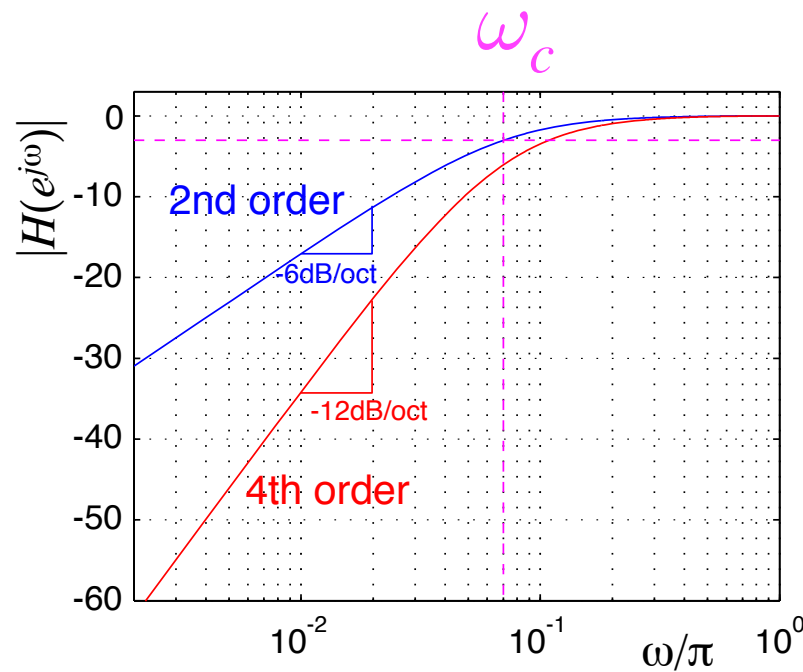


- In general, cascade filters will **not** be optimal (...) for a given order



Cascading Filters

- Cascading filters improves **rolloff** slope:



- But: 3dB cutoff frequency will change (gain at $\omega_c \rightarrow 3N$ dB)

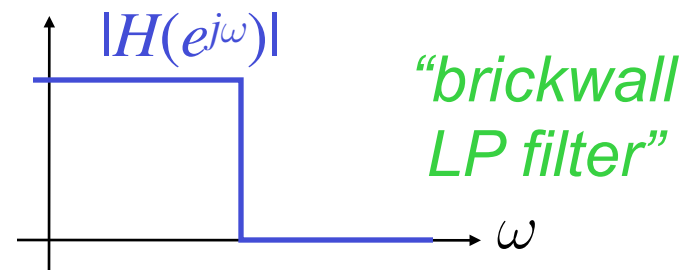


2. Ideal filters

- Typical filter requirements:
 - gain = 1 for wanted parts (**pass band**)
 - gain = 0 for unwanted parts (**stop band**)

- “Ideal” characteristics would be like:

- no phase distortion etc.



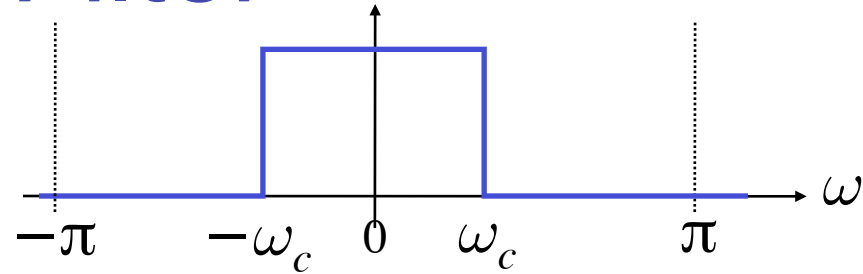
- What is this filter?

- can calculate IR $h[n]$ as IDTFT of ideal response...



Ideal Lowpass Filter

- Given ideal $H(e^{j\omega})$:
(assume $\theta(\omega) = 0$)



$$\begin{aligned}\Rightarrow h[n] &= IDTFT \{ H(e^{j\omega}) \} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega\end{aligned}$$

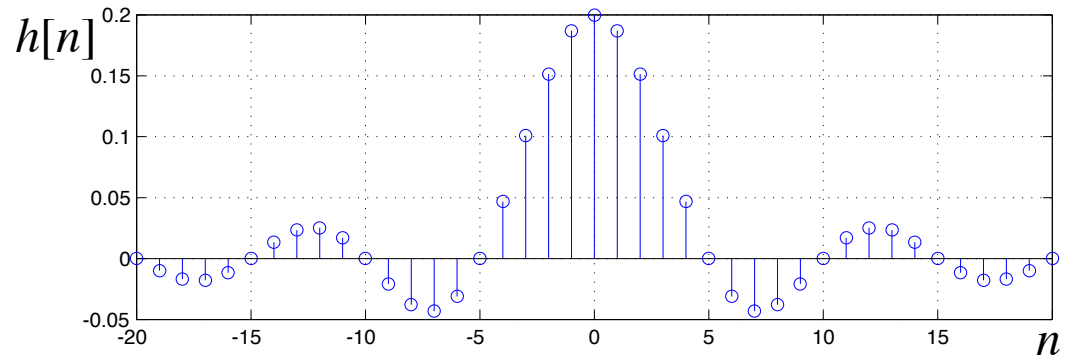
$$\Rightarrow h[n] = \frac{\sin \omega_c n}{\pi n}$$

Ideal lowpass filter



Ideal Lowpass Filter

$$h[n] = \frac{\sin \omega_c n}{\pi n}$$

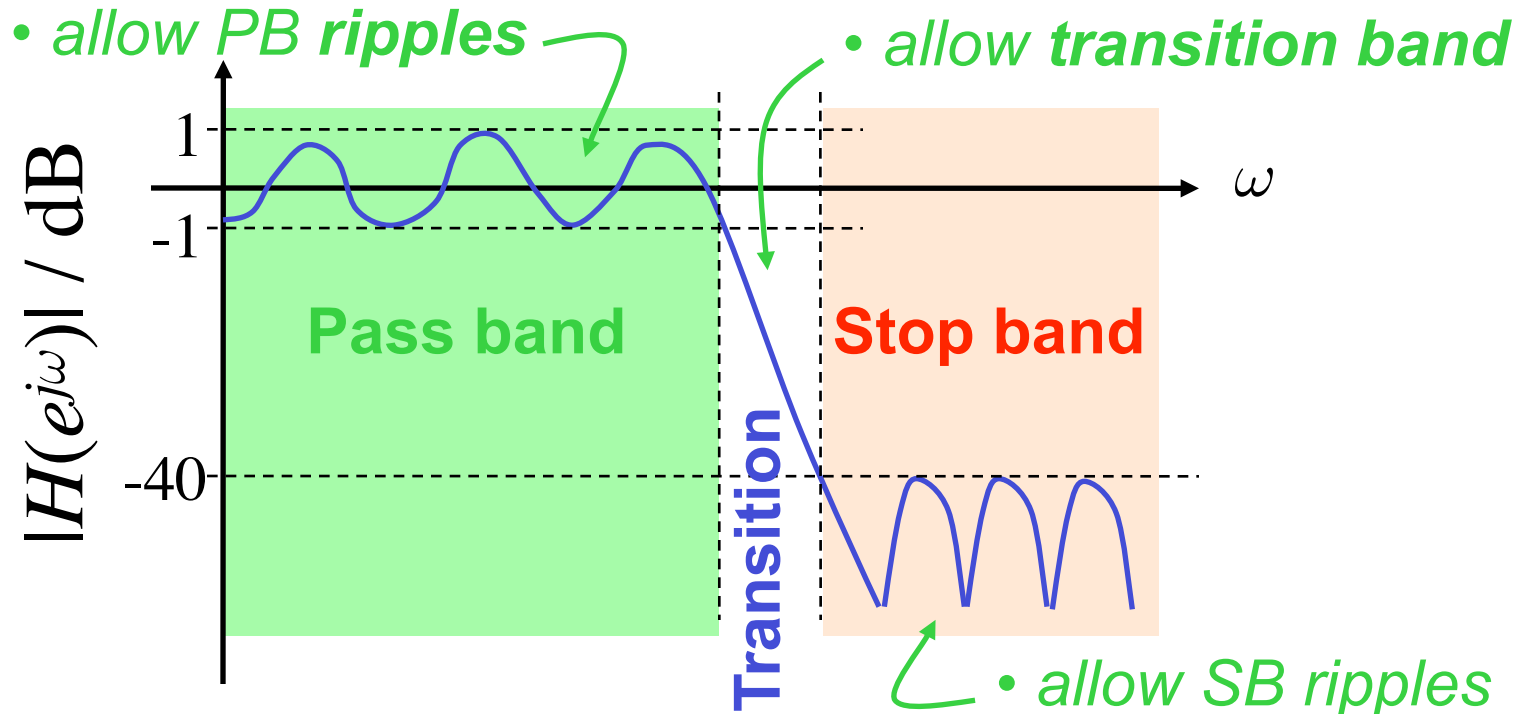


■ Problems!

- doubly infinite ($n = -\infty .. \infty$)
- no rational polynomial \rightarrow very long FIR
- **excellent** *frequency-domain* characteristics
 \leftrightarrow **poor** *time-domain* characteristics
(blurring, ringing – a general problem)



Practical filter specifications



- lower-order realization (less computation)
- better time-domain properties (less ringing)
- easier to design...

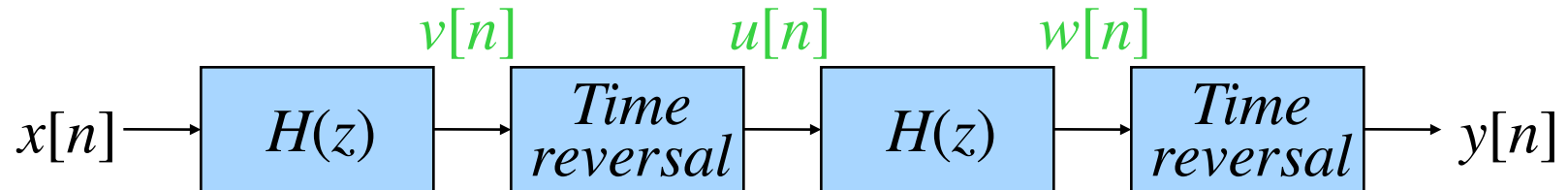


3. Linear-phase Filters

- $|H(e^{j\omega})|$ alone can hide *phase distortion*
 - differing delays for adjacent frequencies can **mangle** the signal
- Prefer filters with a **flat** phase response
e.g. $\theta(\omega) = 0$ **“zero phase filter”**
- A filter with **constant** delay $\tau_p = D$ at all freq's has $\theta(\omega) = -D\omega$ **“linear phase”**
 $\Rightarrow H(e^{j\omega}) = e^{-jD\omega} \tilde{H}(\omega)$ ← *pure-real (zero-phase) portion*
- Linear phase can ‘shift’ to zero phase



Time reversal filtering



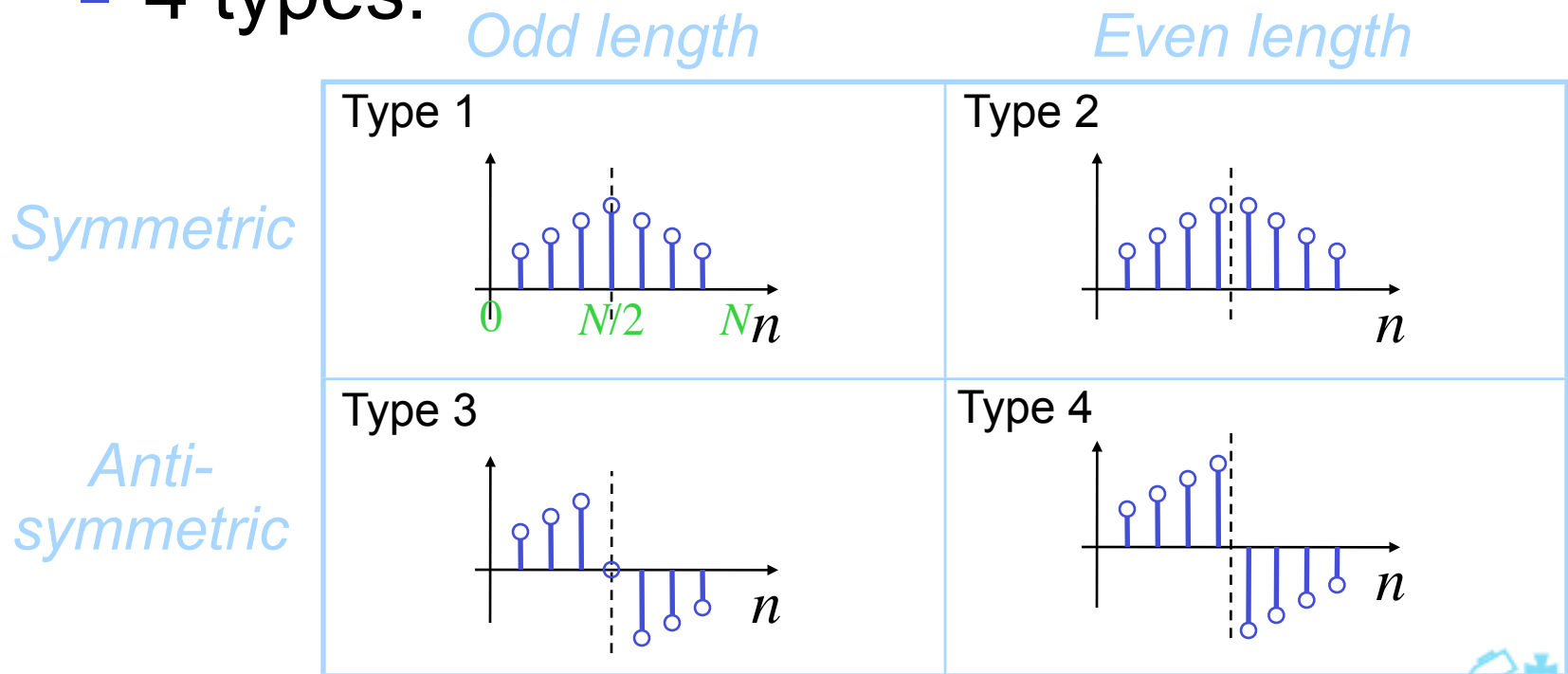
- $v[n] = x[n] \otimes h[n] \rightarrow V(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
- $u[n] = v[-n] \rightarrow U(e^{j\omega}) = V(e^{-j\omega}) = V^*(e^{j\omega})$ *if v real*
- $w[n] = u[n] \otimes h[n] \rightarrow W(e^{j\omega}) = H(e^{j\omega})U(e^{j\omega})$
- $y[n] = w[-n] \rightarrow Y(e^{j\omega}) = W^*(e^{j\omega})$
 $= (H(e^{j\omega})(H(e^{j\omega})X(e^{j\omega}))^*)^*$
 $\rightarrow Y(e^{j\omega}) = X(e^{j\omega})|H(e^{j\omega})|^2$

- Achieves zero-phase result
- **Not causal!** Need whole signal first



Linear Phase FIR filters

- (Anti)Symmetric FIR filters are almost the only way to get zero/linear phase
- 4 types:



Linear Phase FIR: Type 1

- Length L odd \rightarrow order $N = L - 1$ even

- Symmetric $\rightarrow h[n] = h[N - n]$
($h[N/2]$ unique)

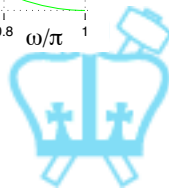
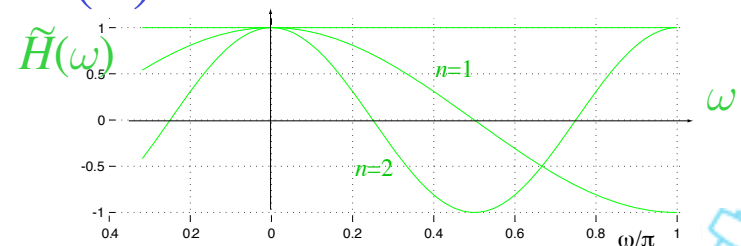
- $H(e^{j\omega}) = \sum_{n=0}^N h[n] e^{-j\omega n}$

$$= e^{-j\omega \frac{N}{2}} \left(h\left[\frac{N}{2}\right] + 2 \sum_{n=1}^{N/2} h\left[\frac{N}{2} - n\right] \cos \omega n \right)$$

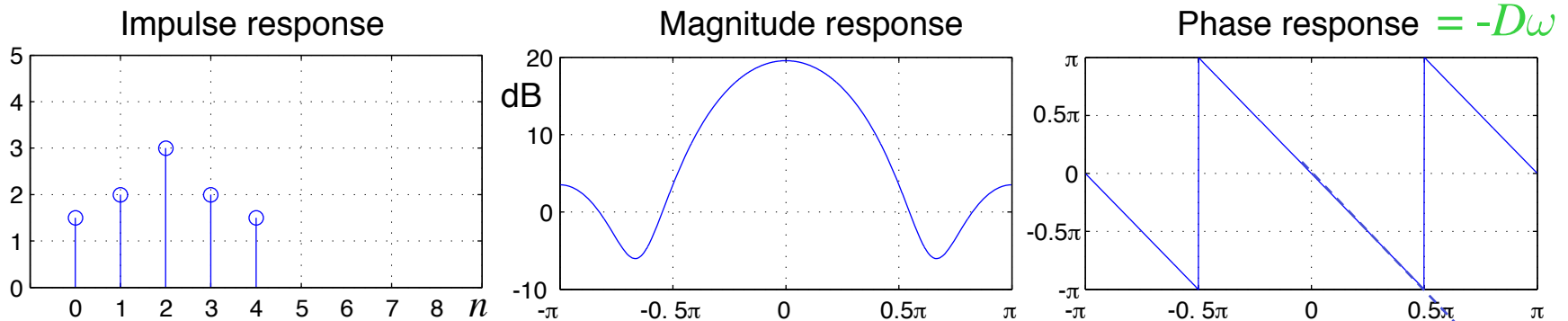
linear phase \rightarrow

$$D = -\theta(\omega)/\omega = N/2$$

pure-real $\tilde{H}(\omega)$ from cosine basis



Linear Phase FIR: Type 1



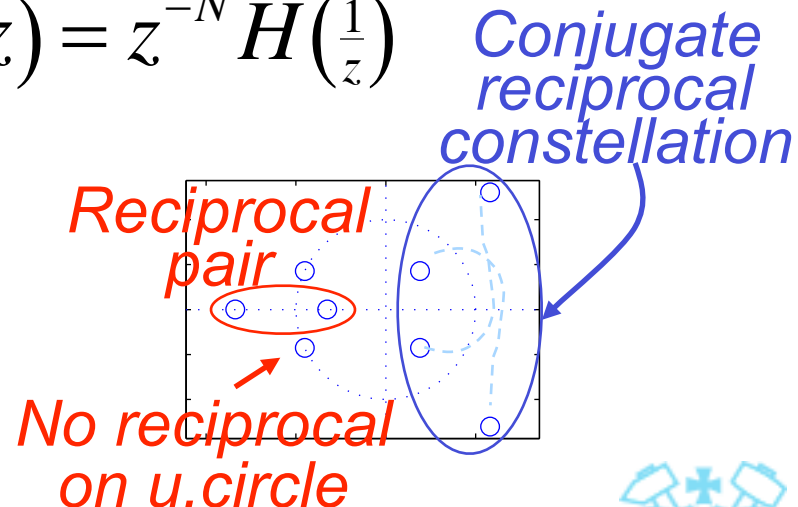
- Where are the N zeros?

$$h[n] = h[N - n] \Rightarrow H(z) = z^{-N} H\left(\frac{1}{z}\right)$$

thus for a zero ζ

$$H(\zeta) = 0 \Rightarrow H\left(\frac{1}{\zeta}\right) = 0$$

Reciprocal zeros
(as well as cplx conj)



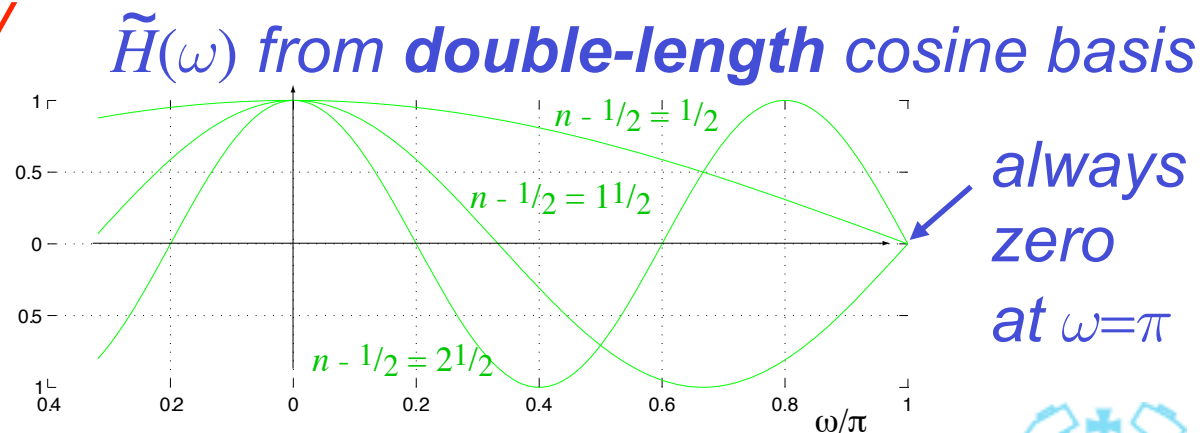
Linear Phase FIR: Type 2

- Length L even \rightarrow order $N = L - 1$ odd

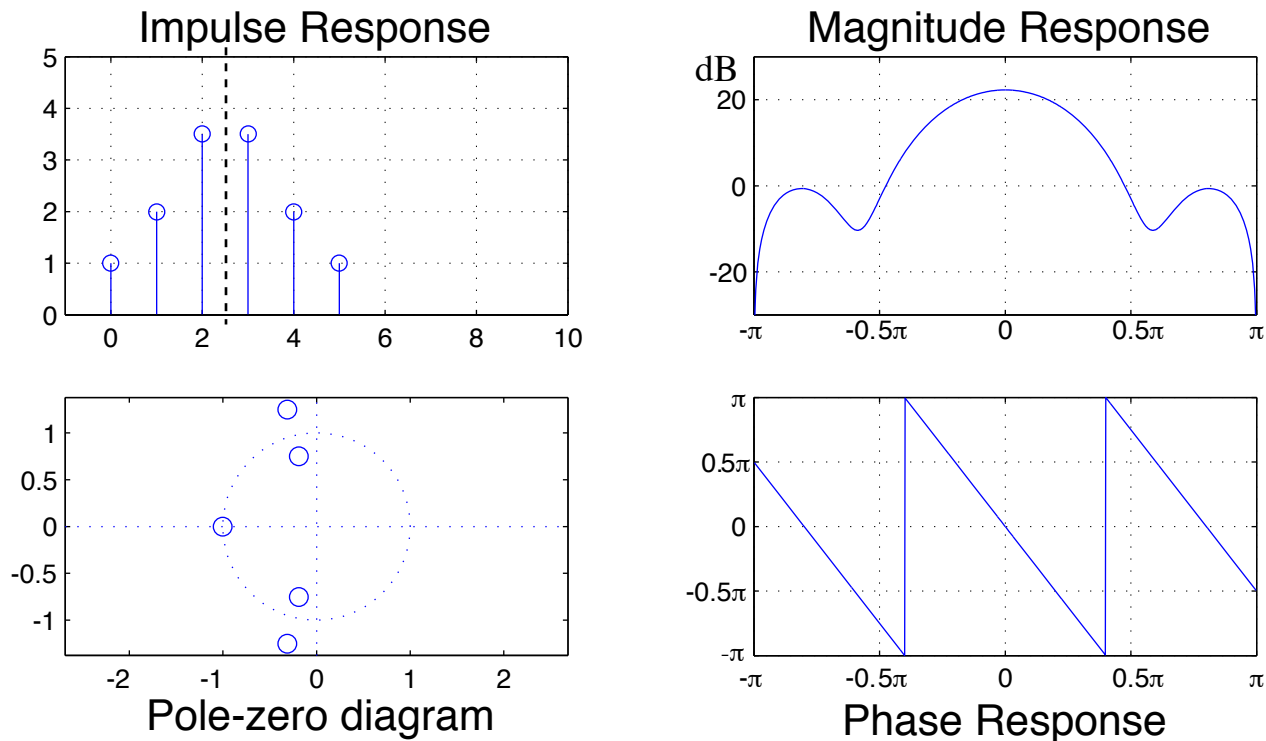
- Symmetric $\rightarrow h[n] = h[N - n]$
(no unique point)

- $$H(e^{j\omega}) = e^{-j\omega \frac{N}{2}} \sum_{n=1}^{(N+1)/2} h\left[\frac{N+1}{2} - n\right] \cos \omega \left(n - \frac{1}{2}\right)$$

Non-integer delay
of $N/2$ samples



Linear Phase FIR: Type 2



- Zeros:** $H(z) = z^{-N} H\left(\frac{1}{z}\right)$
 at $z = -1$, $H(-1) = (-1)^{\overset{\text{odd}}{\uparrow} N} H(-1) \Rightarrow H\left(e^{j\pi}\right) = 0$ LPF-like



Linear Phase FIR: Type 3

- Length L odd \rightarrow order $N = L - 1$ even

- Antisymmetric $\rightarrow h[n] = -h[N - n]$

$$\Rightarrow h[N/2] = -h[N/2] = 0$$

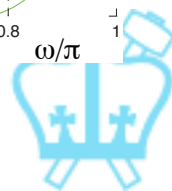
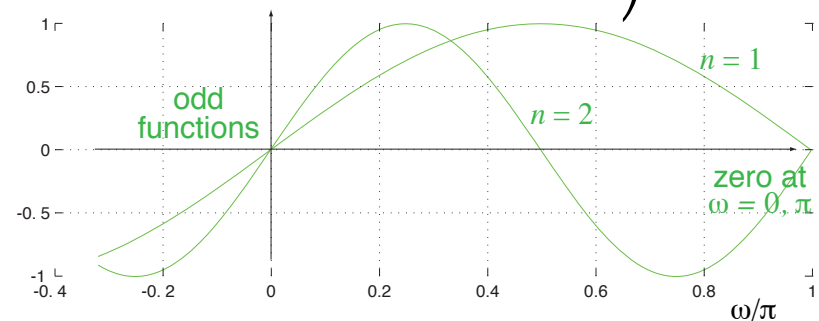
- $$H(e^{j\omega}) = \sum_{n=1}^{N/2} h\left[\frac{N}{2} - n\right] \left(e^{-j\omega\left(\frac{N}{2} - n\right)} - e^{-j\omega\left(\frac{N}{2} + n\right)} \right)$$

$$\Rightarrow je^{-j\omega\frac{N}{2}} \left(2 \sum_{n=1}^{N/2} h\left[\frac{N}{2} - n\right] \sin \omega n \right)$$

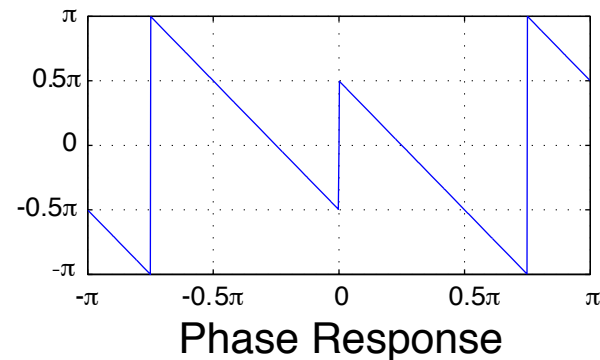
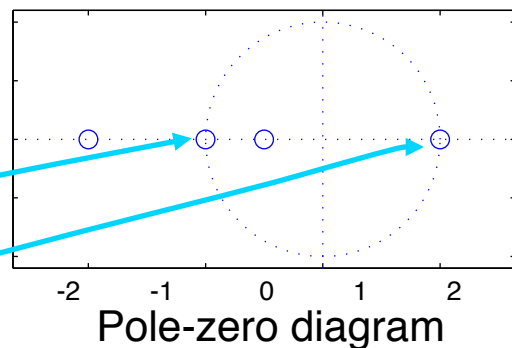
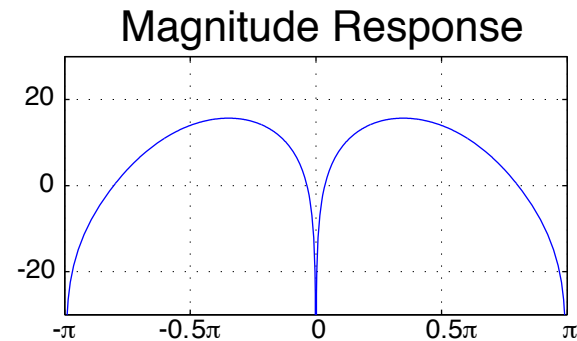
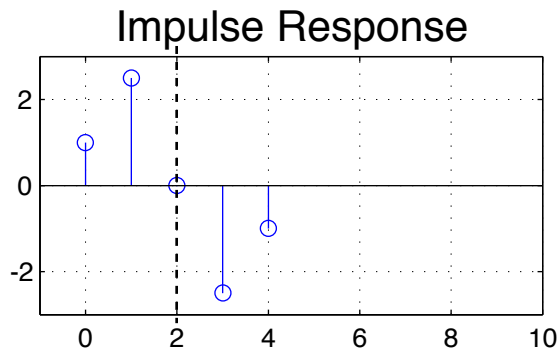
$$\theta(\omega) = \pi/2 - \omega \cdot N/2$$

Antisymmetric \Rightarrow

$\pi/2$ phase shift in addition to linear phase



Linear Phase FIR: Type 3



- Zeros: $H(z) = -z^{-N} H\left(\frac{1}{z}\right)$

$$\Rightarrow H(1) = -H(1) = 0 ; \quad H(-1) = -H(-1) = 0$$



Linear Phase FIR: Type 4

- Length L even \rightarrow order $N = L - 1$ odd

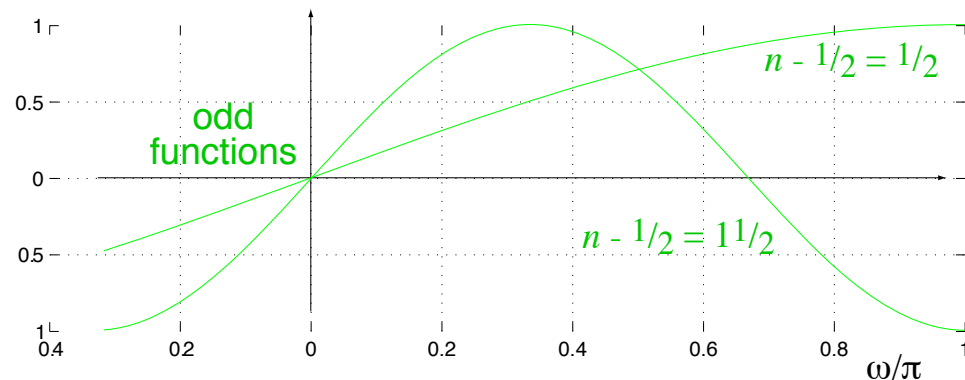
- Antisymmetric $\rightarrow h[n] = -h[N - n]$
(no center point)

- $$H(e^{j\omega}) = je^{-j\omega\frac{N}{2}} 2 \sum_{n=1}^{N/2} h\left[\frac{N+1}{2} - n\right] \sin \omega\left(n - \frac{1}{2}\right)$$

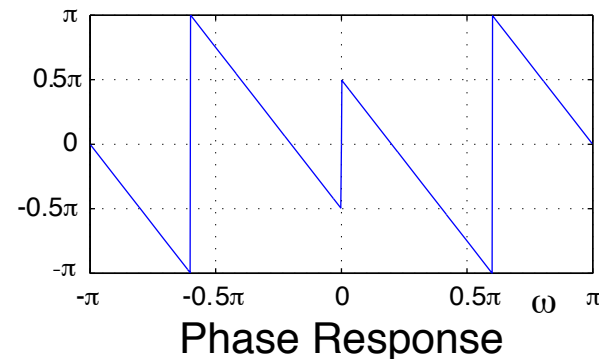
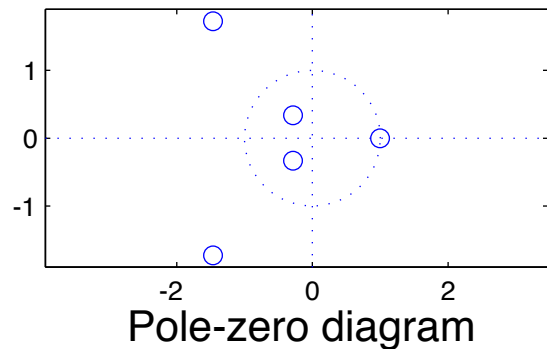
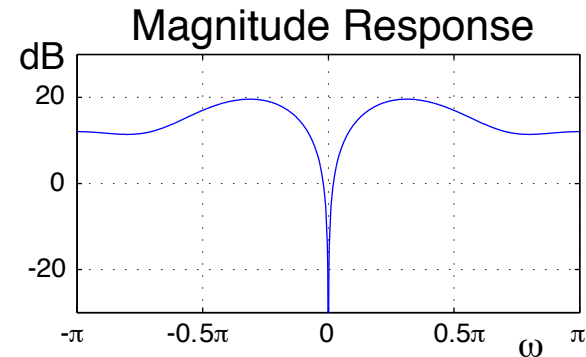
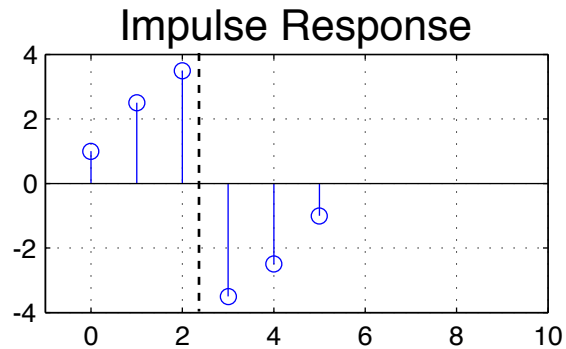
$\pi/2$ offset

fractional-sample delay

offset sine basis



Linear Phase FIR: Type 4



- Zeros: $H(1) = -H(1) = 0$
($H(-1)$ OK because N is odd)



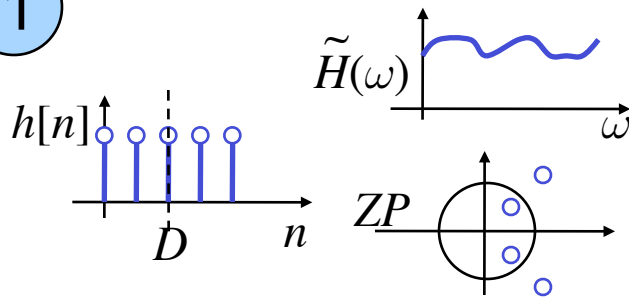
4 Linear Phase FIR Types

Odd length

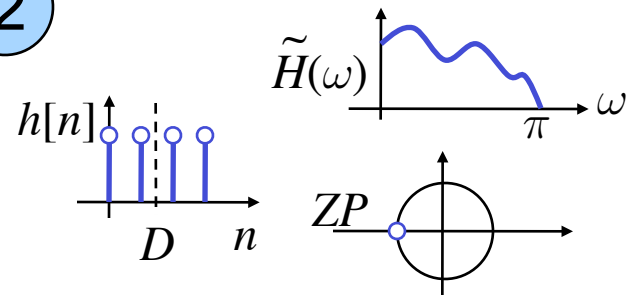
Even length

Symmetric

1

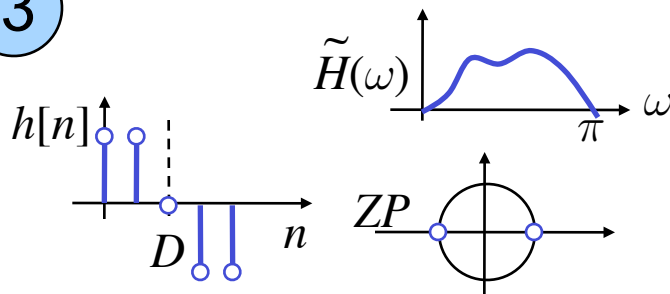


2



Antisymmetric

3



4

