ELEN E4810: Digital Signal Processing
Topic 5:
Transform-Domain Systems

1. Frequency Response (FR)
2. Transfer Function (TF)
3. Phase Delay and Group Delay

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1. Frequency Response (FR)

- Fourier analysis expresses any signal as the sum of sinusoids
  e.g. IDTFT: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$
- Sinusoids are the *eigenfunctions* of LSI systems (only scaled, not ‘changed’)
- Knowing the scaling for every sinusoid fully describes system behavior

→ **frequency response** *describes how a system affects each pure frequency*

Sinusoids as Eigenfunctions

- IR $h[n]$ completely describes LSI system:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] \circledast h[n] = \sum\nolimits_{\forall m} h[m]x[n-m]$$

- Complex sinusoid input i.e. $x[n] = e^{j\omega_0 n}$

$$\Rightarrow y[n] = \sum\nolimits_m h[m]e^{j\omega_0 (n-m)}$$

$$= \underline{\sum\nolimits_m h[m]e^{-j\omega_0 m}} \cdot \underline{e^{j\omega_0 n}}$$

$H(e^{j\omega})$
$= |H(e^{j\omega})|e^{j\theta(\omega)}$

$$\Rightarrow y[n] = H\left(e^{j\omega_0}\right) \cdot x[n] = \left|H\left(e^{j\omega_0}\right)\right| \cdot e^{j(\omega_0 n + \theta(\omega_0))}$$

- Output is sinusoid scaled by FT at ω_0

System Response from $H(e^{j\omega})$

- If $x[n]$ is a complex sinusoid at ω_0

  then the output of a system with IR $h[n]$ is the same sinusoid scaled by $|H(e^{j\omega_0})|$

  and phase-shifted by $\arg\{H(e^{j\omega_0})\} = \theta(\omega_0)$

  where $H(e^{j\omega}) = \mathrm{DTFT}\{h[n]\}$

(Any signal can be expressed as sines...)

- $|H(e^{j\omega})|$ "magnitude response" → gain
- $\arg\{H(e^{j\omega})\}$ "phase resp." → phase shift

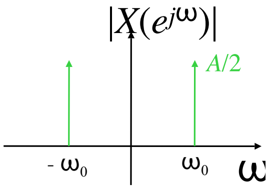
Real Sinusoids

- In practice signals are real e.g.

$$x[n] = A\cos(\omega_0 n + \phi)$$

$$= \frac{A}{2}\left(e^{j(\omega_0 n+\phi)} + e^{-j(\omega_0 n+\phi)}\right)$$

$$= \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

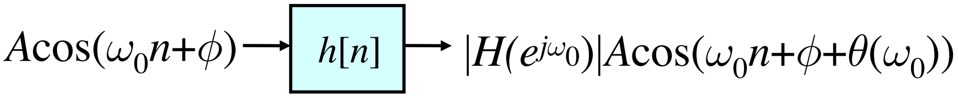


$$\Rightarrow y[n] = \frac{A}{2}e^{j\phi}H\left(e^{j\omega_0}\right)e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}H\left(e^{-j\omega_0}\right)e^{-j\omega_0 n}$$

- Real $h[n]$ → $H(e^{-j\omega}) = H^*(e^{j\omega}) = |H(e^{j\omega})|e^{-j\theta(\omega)}$

$$\Rightarrow y[n] = A\left|H\left(e^{j\omega_0}\right)\right|\cos\left(\omega_0 n + \phi + \theta(\omega_0)\right)$$

Real Sinusoids



$A\cos(\omega_0 n+\phi)$ → $h[n]$ → $|H(e^{j\omega_0})|A\cos(\omega_0 n+\phi+\theta(\omega_0))$

- A real sinusoid of frequency ω_0 passed through an LSI system with a real impulse response $h[n]$ has its gain modified by $|H(e^{j\omega_0})|$ and its phase shifted by $\theta(\omega_0)$.

Transient / Steady State

- Most signals start at a finite time e.g.
  $x[n] = e^{j\omega_0 n}\mu[n]$ What is the effect?

$$\begin{aligned} y[n] &= h[n] \circledast x[n] = \sum_{m=-\infty}^{n} h[m]e^{j\omega_0(n-m)} \\ &= \sum_{m=-\infty}^{\infty} h[m]e^{j\omega_0(n-m)} - \sum_{m=n+1}^{\infty} h[m]e^{j\omega_0(n-m)} \\ &= \underline{H(e^{j\omega_0})e^{j\omega_0 n}} - \underline{\left(\sum_{m=n+1}^{\infty} h[m]e^{-j\omega_0 m}\right)e^{j\omega_0 n}} \end{aligned}$$

Steady state
- same as with pure sine input

Transient response
- consequence of gating

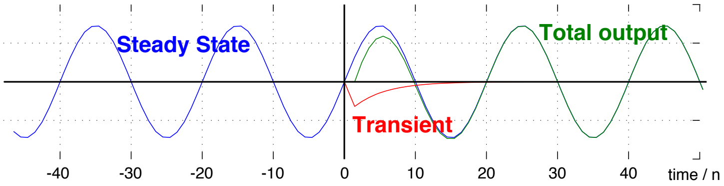
Transient / Steady State

- $x[n] = e^{j\omega_0 n}\mu[n]$

transient

$$\Rightarrow y[n] = H(e^{j\omega_0})e^{j\omega_0 n} - \left(\sum_{m=n+1}^{\infty} h[m]e^{-j\omega_0 m}\right)e^{j\omega_0 n}$$

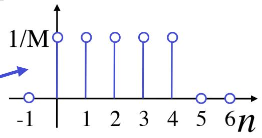
  - FT of IR $h[n]$'s *tail* from time n onwards
  - zero for FIR $h[n]$ for $n \geq N$
  - tends to zero with large n for any 'stable' IR



FR example

- MA filter $y[n] = \frac{1}{M}\sum_{\ell=0}^{M-1} x[n-\ell]$

$$= x[n] \circledast h[n]$$



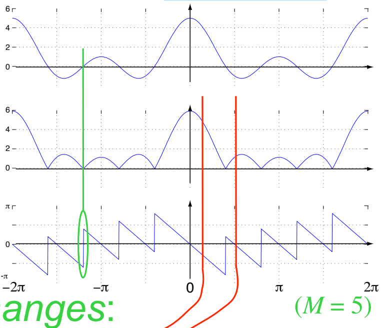
$$\Rightarrow \quad H\left(e^{j\omega}\right) = DTFT\{h(n)\}$$

$$= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \tfrac{1}{M}\sum_{n=0}^{M-1} e^{-j\omega n}$$

$$= \frac{1}{M}\frac{1-e^{-j\omega M}}{1-e^{-j\omega}} = \frac{1}{M}e^{-j\omega(M-1)/2}\frac{\sin(M\omega/2)}{\sin(\omega/2)}$$

FR example

- MA filter: $H\left(e^{j\omega}\right) = \tfrac{1}{M}e^{-j\omega(M-1)/2}\frac{\sin(M\omega/2)}{\sin(\omega/2)}$



$$\Rightarrow \left|H\left(e^{j\omega}\right)\right| = \left|\frac{1}{M}\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right|$$

$$\theta(\omega) = \frac{-(M-1)\omega}{2} + \pi \cdot r$$

(jumps at sign changes: $r = \lfloor M\omega/2\pi \rfloor$)

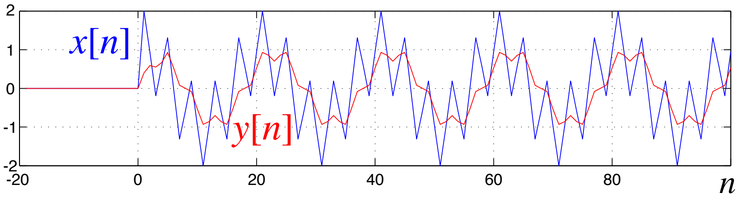
- Response to $x[n] = e^{j\omega_0 n} + e^{j\omega_1 n}$ ...

FR example

- MA filter
- input $x[n]=e^{j\omega_0 n}+e^{j\omega_1 n}$

  $\omega_0=0.1\pi \rightarrow H(e^{j\omega_0}) \sim \ 4/5\ e^{j\phi_0}$

  $\omega_1=0.5\pi \rightarrow H(e^{j\omega_1}) \sim \ -1/5\ e^{j\phi_1}$
- output $y[n]=H(e^{j\omega_0})e^{j\omega_0 n}+H(e^{j\omega_1})e^{j\omega_1 n}$



2. Transfer Function (TF)

Linking LCCDE, ZT & Freq. Resp...

- LCCDE: $\sum_{k=0}^{N} d_k y[n-k]=\sum_{k=0}^{M} p_k x[n-k]$
- Take ZT: $\sum_k d_k z^{-k} Y(z)=\sum_k p_k z^{-k} X(z)$
- Hence: $Y(z)=\dfrac{\sum_k p_k z^{-k}}{\sum_k d_k z^{-k}}X(z)$
- or: $Y(z)=H(z)X(z)$

Transfer function

H(z)

Transfer Function (TF)

- Alternatively, $y[n] = h[n] \circledast x[n]$

  ZT → $Y(z) = H(z)X(z)$

- Note: *same* $H(z) = \begin{cases} \frac{\sum p_k z^{-k}}{\sum d_h z^{-k}} & \text{... if system has DE form} \\ \sum_n h[n] z^{-n} & \text{... from IR} \end{cases}$

- e.g. FIR filter, $h[n] = \{h_0, h_1, \ldots h_{M\text{-}1}\}$ ✓

$\Rightarrow p_k = h_k$, $d_0 = 1$, DE is $1 \cdot y[n] = \sum_{k=0}^{M-1} h_k x[n-k]$

Transfer Function (TF)

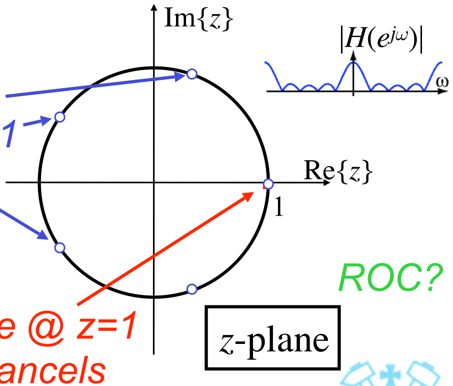
- Hence, MA filter:

$$y[n] = \frac{1}{M} \sum_{\ell=0}^{M-1} x[n-\ell] \quad \Rightarrow h[n] = \begin{cases} \frac{1}{M} & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

$$H(z) = \tfrac{1}{M} \sum_{n=0}^{M-1} z^{-n}$$

$$= \frac{1 - z^{-M}}{M\left(1 - z^{-1}\right)}$$

$$= \frac{z^M - 1}{M \cdot z^{M-1}(z-1)}$$



z^M=1 *i.e.*

M roots of 1

@ $z = e^{jr\pi/M}$

pole @ z=1 cancels

ROC?

TF example

- $y[n] = x[n\text{-}1] - 1.2\,x[n\text{-}2] + x[n\text{-}3]$
  $+ 1.3\,y[n\text{-}1] - 1.04\,y[n\text{-}2] + 0.222\,y[n\text{-}3]$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - 1.2z^{-2} + z^{-3}}{1 - 1.3z^{-1} + 1.04z^{-2} - 0.222z^{-3}}$$

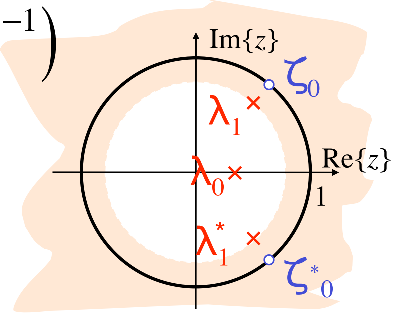
- factorize:

$$H(z) = \frac{z^{-1}\left(1-\zeta_0 z^{-1}\right)\left(1-\zeta_0^* z^{-1}\right)}{\left(1-\lambda_0 z^{-1}\right)\left(1-\lambda_1 z^{-1}\right)\left(1-\lambda_1^* z^{-1}\right)}$$

$\zeta_0 = 0.6+j0.8$
$\lambda_0 = 0.3$
$\lambda_1 = 0.5+j0.7$
→ ...

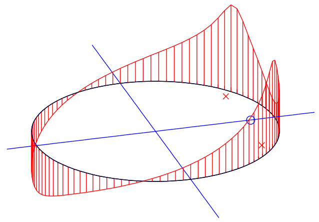
TF example

$$H(z) = \frac{z^{-1}\left(1-\zeta_0 z^{-1}\right)\left(1-\zeta_0^* z^{-1}\right)}{\left(1-\lambda_0 z^{-1}\right)\left(1-\lambda_1 z^{-1}\right)\left(1-\lambda_1^* z^{-1}\right)}$$



- Poles λ_i → ROC
  - *causal* → ROC is $|z| > \max|\lambda_i|$
  - includes u.circle → *stable*

TF → FR



- DTFT $H(e^{j\omega})$ = ZT $H(z)|_{z=e^{j\omega}}$

  i.e. Frequency Response is Transfer Function eval'd on Unit Circle

factor:

$$H(z)=\frac{p_0\prod_{k=1}^{M}\left(1-\zeta_k z^{-1}\right)}{d_0\prod_{k=1}^{N}\left(1-\lambda_k z^{-1}\right)}=\frac{p_0 z^{-M}\prod_{k=1}^{M}(z-\zeta_k)}{d_0 z^{-N}\prod_{k=1}^{N}(z-\lambda_k)}$$

$$\Rightarrow H\left(e^{j\omega}\right)=\frac{p_0}{d_0}e^{j\omega(N-M)}\frac{\prod_{k=1}^{M}\left(e^{j\omega}-\zeta_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega}-\lambda_k\right)}$$

TF → FR

$$H\left(e^{j\omega}\right)=\frac{p_0}{d_0}e^{j\omega(N-M)}\frac{\prod_{k=1}^{M}\left(e^{j\omega}-\zeta_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega}-\lambda_k\right)}$$

ζ_i, λ_i are TF roots on z-plane

$$\Rightarrow \left|H(e^{j\omega})\right| = \left|\frac{p_0}{d_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega}-\zeta_k\right|}{\prod_{k=1}^{M}\left|e^{j\omega}-\lambda_k\right|}$$

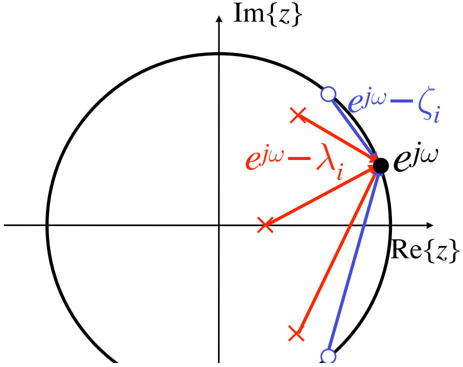
Magnitude response

$$\theta(\omega)=\arg\left\{\frac{p_0}{d_0}\right\}+\omega\cdot(N-M)$$
$$+\sum_{k=1}^{M}\arg\left\{e^{j\omega}-\zeta_k\right\}-\sum_{k=1}^{N}\arg\left\{e^{j\omega}-\lambda_k\right\}$$

Phase response

FR: Geometric Interpretation

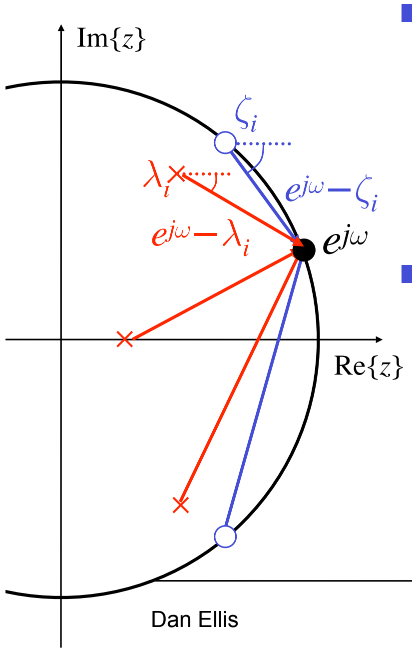
- Have $$H\left(e^{j\omega}\right)=\underbrace{\frac{p_0}{d_0}\,e^{j\omega(N-M)}}_{\text{Constant/ linear part}}\;\underbrace{\frac{\prod_{k=1}^{M}\left(e^{j\omega}-\zeta_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega}-\lambda_k\right)}}_{\text{Product/ratio of terms related to poles/zeros}}$$
- On z-plane:



Each ($e^{j\omega}$ - ν) term corresponds to a vector from pole/zero ν to point $e^{j\omega}$ on the unit circle

Overall FR is *product/ratio* of all these vectors

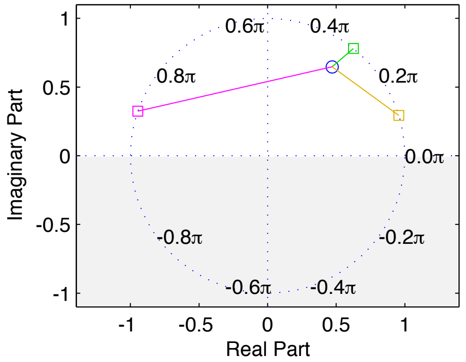
FR: Geometric Interpretation

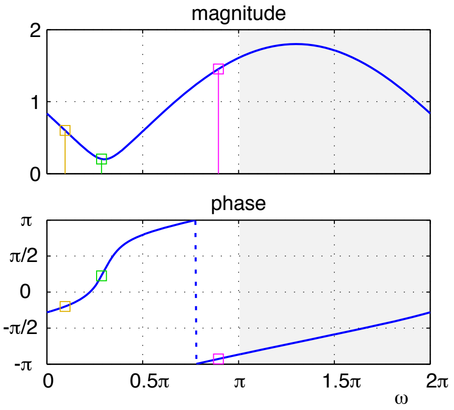


- **Magnitude** $|H(e^{j\omega})|$ is product of lengths of vectors from zeros divided by product of lengths of vectors from poles
- **Phase** $\theta(\omega)$ is sum of angles of vectors from zeros minus sum of angles of vectors from poles

FR: Geometric Interpretation

- Magnitude and phase of a single zero:



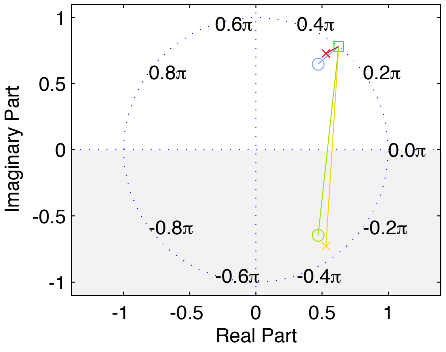


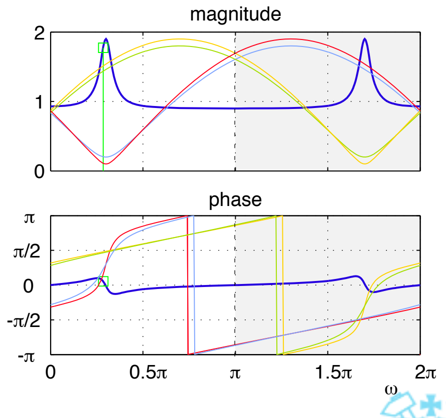
- Pole is reciprocal mag. & negated phase

FR: Geometric Interpretation

- Multiple poles, zeros:

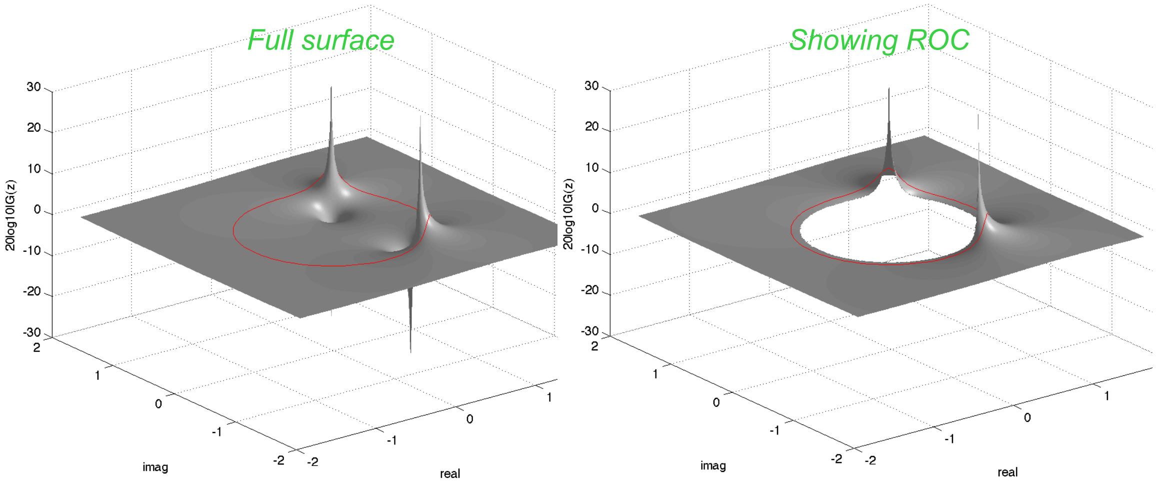
$$H(z) = \frac{\left(z - 0.8e^{j0.3\pi}\right)\left(z - 0.8e^{-j0.3\pi}\right)}{\left(z - 0.9e^{j0.3\pi}\right)\left(z - 0.9e^{-j0.3\pi}\right)}$$





Geom. Interp. vs. 3D surface

- 3D magnitude surface for same system



Geom. Interp: Observations

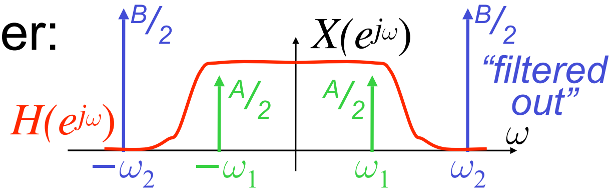
- Roots near unit circle
  → rapid changes in magnitude & phase
  - zeros cause mag. minima (= 0 → on u.c.)
  - poles cause mag. peaks (→ 1÷0=∞ at u.c.)
  - rapid change in relative angle → phase
- Pole and zero ‘near’ each other
  cancel out when seen from ‘afar’;
  affect behavior when $z = e^{j\omega}$ gets ‘close’

Filtering

- Idea: Separate information in frequency with constructed $H(e^{j\omega})$
- e.g. $x[n] = \underbrace{A\cos(\omega_1 n)}_{\text{interested in this part}} + \underbrace{B\cos(\omega_2 n)}_{\text{don't care about this part}}$
- Construct a filter:

  $|H(e^{j\omega_1})| \sim 1$

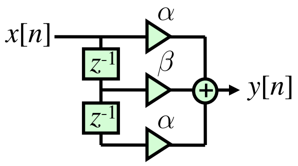
  $|H(e^{j\omega_2})| \sim 0$

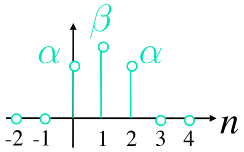


- Then $y[n] = h[n] \circledast x[n] \approx A\cos(\omega_1 n + \theta(\omega_1))$

Filtering example

- Consider filter 'family': 3 pt FIR filters with $h[n]$ = $\{\alpha\ \beta\ \alpha\}$





- Frequency Response:

$$H\left(e^{j\omega}\right) = \sum\nolimits_{\forall n} h[n]e^{-j\omega n} = \alpha + \beta e^{-j\omega} + \alpha e^{-2j\omega}$$

$$= e^{-j\omega}\left(\beta + \alpha\left(e^{j\omega} + e^{-j\omega}\right)\right) = e^{-j\omega}(\beta + 2\alpha\cos\omega)$$

$$\Rightarrow \left|H\left(e^{j\omega}\right)\right| = |\beta + 2\alpha\cos\omega|$$

can set α and β to obtain desired $|H(e^{j\omega})|$ ...

Filtering example (cont’d)

- $h[n] = \{\alpha\ \beta\ \alpha\} \quad \Rightarrow \left|H\left(e^{j\omega}\right)\right| = \left|\beta + 2\alpha\cos\omega\right|$
- Consider input as mix of sinusoids at $\omega_1 = 0.1$ rad/samp and $\omega_2 = 0.4$ rad/samp

want to remove
i.e. make $H(e^{j\omega_2}) = 0$

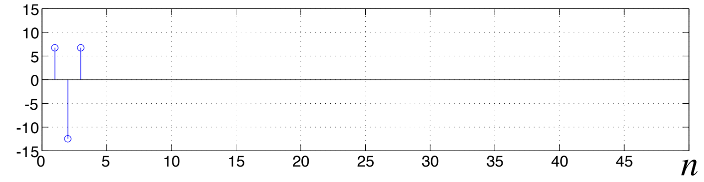
- Solve $\left|H(e^{j\omega})\right| = |\beta + 2\alpha\cos\omega|$

$$= \begin{cases} 1 & \omega = \omega_1 = 0.1 \\ 0 & \omega = \omega_2 = 0.4 \end{cases}$$

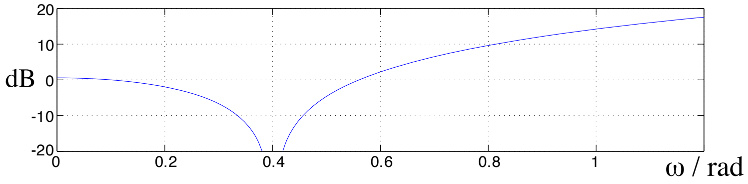
$\Rightarrow \beta = \text{-}12.46,\ \alpha = 6.76\ \ \ldots$

Filtering example (cont’d)

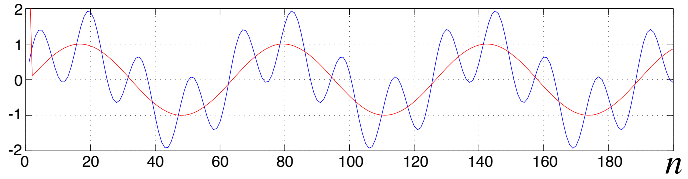
- Filter IR



- Freq. resp



- input/ output



3. Phase- and group-delay

- For sinusoidal input $x[n] = \cos\omega_0 n$,

  we saw $y[n] = \underbrace{\left|H\left(e^{j\omega_0}\right)\right|}_{\text{gain}} \cos\left(\omega_0 n + \underbrace{\theta(\omega_0)}_{\text{phase shift or \textbf{time} shift}}\right)$

- i.e. $\cos\left(\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right)\right)$

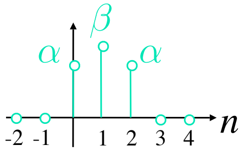
  or $\cos\left(\omega_0\left(n - \tau_p(\omega_0)\right)\right)$

  subtraction so positive τ_p means delay (causal)

- where $\tau_p(\omega) = \frac{-\theta(\omega)}{\omega}$ is **phase delay**

Phase delay example

- For our 3pt filter:



$$H\left(e^{j\omega}\right) = e^{-j\omega}\left(\beta + 2\alpha\cos\omega\right)$$

$$\Rightarrow \theta(\omega) = -\omega$$

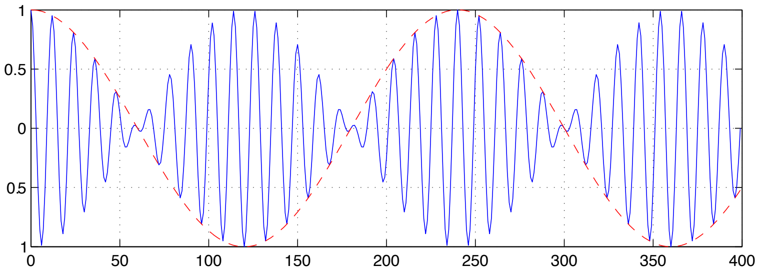
$$\Rightarrow \tau_p(\omega) = -\left(\frac{-\omega}{\omega}\right) = +1$$

- i.e. 1 sample delay (at all frequencies) (as observed)

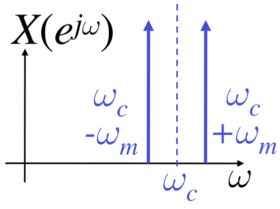
Group Delay

- Consider a modulated carrier
  e.g. $x[n] = A[n]\cdot\cos(\omega_c n)$

  with $A[n] = A\cos(\omega_m n)$ and $\omega_m << \omega_c$



Group Delay



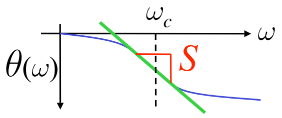
- So: $x[n] = A\cos(\omega_m n)\cdot\cos(\omega_c n)$
  $$= \tfrac{A}{2}\left[\cos(\omega_c - \omega_m)n + \cos(\omega_c + \omega_m)n\right]$$
- Now: $y[n] = h[n] \circledast x[n]$
  $$= \frac{A}{2}\begin{pmatrix} H\left(e^{j(\omega_c-\omega_m)}\right)\cos(\omega_c - \omega_m)n \\ +H\left(e^{j(\omega_c+\omega_m)}\right)\cos(\omega_c + \omega_m)n \end{pmatrix}$$
- Assume $|H(e^{j\omega})| \sim 1$ around $\omega_c \pm \omega_m$
  but $\theta(\omega_c - \omega_m) = \theta_l$; $\theta(\omega_c + \omega_m) = \theta_u$ ...

Group Delay

$$y[n] = \tfrac{A}{2}\begin{pmatrix}\cos[(\omega_c - \omega_m)n + \theta_l] \\ +\cos[(\omega_c + \omega_m)n + \theta_u]\end{pmatrix}$$

$$= A\cos\left(\omega_c n + \underbrace{\frac{\theta_u + \theta_l}{2}}_{\textit{phase shift of carrier}}\right)\cdot\cos\left(\omega_m n + \underbrace{\frac{\theta_u - \theta_l}{2}}_{\textit{phase shift of envelope}}\right)$$

Group Delay

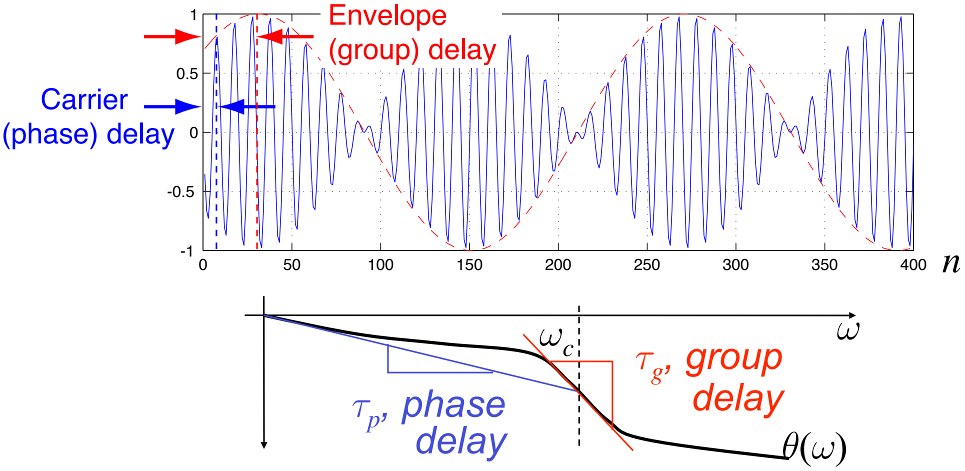


- If $\theta(\omega_c)$ is locally linear i.e. $\theta(\omega_c+\Delta\omega) = \theta(\omega_c) + S\Delta\omega$, $S = \left.\dfrac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c}$
- Then carrier *phase shift* $\dfrac{\theta_l + \theta_u}{2} = \theta(\omega_c)$

  so carrier delay $-\dfrac{\theta(\omega_c)}{\omega_c} = \tau_p$, phase delay
- Envelope *phase shift* $\dfrac{\theta_u - \theta_l}{2} = \omega_m \cdot S$

→ delay $\tau_g(\omega_c) = -\left.\dfrac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c}$ **group delay**

Group Delay



- If $\theta(\omega)$ is not linear around ω_c, $A[n]$ suffers "phase distortion" → correction...