2012-10-24 Dan Ellis <dpwe@ee.columbia.edu>

1. (a) i. Taking the output of the summation node, we have:

$$y[n] = \alpha y[n-1] + x[n] + \alpha x[n-1] + \alpha^2 x[n-2] \tag{1}$$

We make this into canonical difference equation form by gathering all the y terms on one side:

$$y[n] - \alpha y[n-1] = x[n] + \alpha x[n-1] + \alpha^2 x[n-2] \tag{2}$$

ii. From the previous step, taking the z-transform is straightforward:

$$Y(z) - \alpha z^{-1} Y(z) = X(z) + \alpha z^{-1} X(z) + \alpha^2 z^{-2} X(z) \tag{3}$$

Thus,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \alpha z^{-1} + \alpha^2 z^{-2}}{1 - \alpha z^{-1}} \tag{4}$$

Since the system is causal as defined, and it has a pole at $z = \alpha$, the region of convergence is $|z| > |\alpha|$.

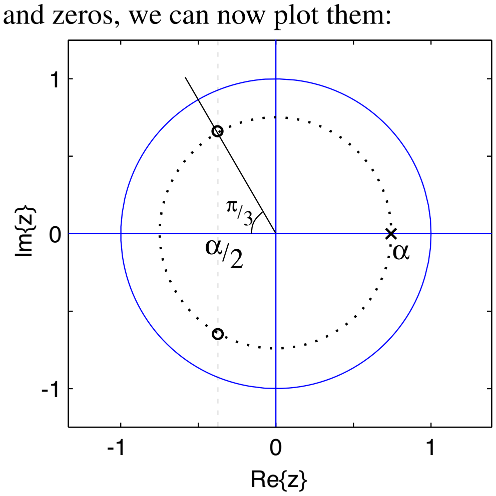
iii. The values of z where $|H(z)| = 0$ are of course the zeros, the roots of the numerator. These are the solutions to $z^2 + \alpha z + \alpha^2 = 0$. The quadratic solution gives us:

$$\zeta = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\alpha^2}}{2} = -\frac{\alpha}{2} \pm j\frac{\alpha}{2}\sqrt{3} \tag{5}$$

This is a pair of conjugate complex zeros whose magnitudes (which is what the question asks for) are both $\sqrt{\mathbb{R}^2 + \mathbb{I}^2} = \sqrt{(-\frac{\alpha}{2})^2 + (\frac{\alpha}{2}\sqrt{3})^2} = \alpha$.

The values of z where $|H(z)| = \frac{1}{0}$ are the poles, which by inspection is $z = \alpha$ (a single real pole). Thus, its magnitude is also α.

iv. Having found the poles and zeros, we can now plot them:



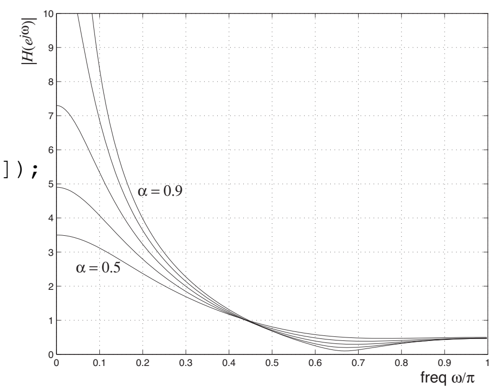
If we remember our trigonometry, we will be able to spot that the angle that the zeros make with the real axis is $\cos^{-1} 0.5 = \pi/3$.

(b) All I'm looking for here is the qualitative observation: "The pole and the zeros all have magnitude equal to α" – which is to say, they all lie on a circle of radius α.

(c) Here is the promised sketching. The pole on the real axis will give a peak at d.c., and the conjugate zeros will give a local dip in the high-frequency region (at least for α close enough to 1). As α gets closer to 1, the dips will become more pronounced (note, this system does not involve any gain normalization, so the pole gain simply gets larger as it gets closer to the unit circle). For $\alpha = 1$, we have a pole on the unit circle, which is an unstable filter, so should not be plotted.

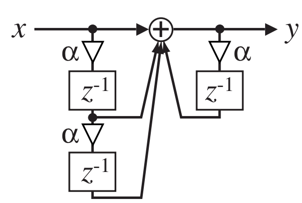
Here are some actual curves plotted by Matlab:

```
>> for alpha = 0.5:0.1:0.9;
        [H,W] = freqz([1 alpha alpha^2], [1 -alpha]);
        plot(W/pi, abs(H));
        hold on;
   end
```



A good sketch will indicate the magnitudes at $\omega = 0, \pi$, the frequency of the minimum, and maybe its magnitude. At d.c., $e^{j\omega} = 1$, so $|H(e^{j\omega})| = (1 + \alpha + \alpha^2)/(1 - \alpha) = 1.75/0.5 = 3.5$ for $\alpha = 0.5$, $1.96/0.4 = 4.9$ for $\alpha = 0.6$ etc. At $\omega = \pi$, $e^{j\omega} = -1$, so $H(e^{j\omega}) = (1 - \alpha + \alpha^2)/(1 + \alpha) = 0.75/1.5 = 0.5$ for $\alpha = 0.5$, $0.76/1.6 = 0.475$ for $\alpha = 0.6$, etc. The minimum is tending towards $\omega = 2\pi/3$; by considering the lengths of the three "vectors", $|e^{j\omega} - \zeta|$, $|e^{j\omega} - \zeta^*|$, and $1/|e^{j\omega} - \lambda|$, the magnitude of the short one is $1 - \alpha$, and the other two (one to the further zero and one to the pole), are, by symmetry, the same length, so their influence cancels out and the magnitude at the minimum is $1 - \alpha$.

(d) We have $H(z) = \frac{1+\alpha z^{-1}+\alpha^2 z^{-2}}{1-\alpha z^{-1}}$. The question is asking us to rewrite this using $u = z/\alpha$, which is simply $H(z) = \frac{1+u^{-1}+u^{-2}}{1-u^{-1}}|_{u=z/\alpha}$. This suggests a way of refactoring the block diagram to include the α terms with each delay element, e.g.:



The point of this is to get at the idea that $H(z) = H(\alpha u)$, where $H(u)$ is the system with all roots on the unit circle; $H(\alpha u)$ is the same surface with a uniform scaling of the u-plane by a factor of $1/\alpha$. But you didn't need to notice or describe this to get full marks.

(e) We could solve for the impulse response by solving the LCCDE with zero initial conditions. It seems easier to me to do it by considering the impulse responses of the feed-forward (zeros) and feed-back (poles), and convolving them. By inspection, the feed-forward part has a simple three-point impulse response $h_z[n] = \{1, \alpha, \alpha^2\}$; the single-delay feedback evidently has an infinite, exponentially-decaying impulse response $h_p[n] = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, ...\}$. Thus, their convolution consists of three instances of h_p, delayed and scaled by the values in h_z, i.e.

$$\begin{aligned} h[n] =& h_z[n] \circledast h_p[n] \\ =& h_p[n] + \alpha h_p[n-1] + \alpha^2 h_p[n-2] \\ =& \delta[n] + 2\alpha\delta[n-1] + 3\sum_{k=2}^{\infty} \alpha^k \delta[n-k] \end{aligned} \tag{6}$$

(f) Since we are asked for the 4-point DFT, the simplest thing is just to evaluate $H(e^j\omega)$ at the four equal-spaced points around the unit circle, i.e.:

$$\begin{aligned} H[k] =& H(z)|_{z=\{e^{j0}, e^{j\pi/2}, e^{j\pi}, e^{3j\pi/2}\}} \\ =& \left[\frac{1+\alpha z^{-1}+\alpha^2 z^{-2}}{1-\alpha z^{-1}}\right]_{z=\{1,j,-1,-j\}} \\ =& \left\{\frac{1+\alpha+\alpha^2}{1-\alpha}, \frac{1-j\alpha-\alpha^2}{1+j\alpha}, \frac{1-\alpha+\alpha^2}{1+\alpha}, \frac{1+j\alpha-\alpha^2}{1-j\alpha}\right\} \\ =& \left\{\frac{1.75}{0.5}, \frac{0.75-0.5j}{1+0.5j}, \frac{0.75}{1.5}, \frac{0.75+0.5j}{1-0.5j}\right\} \\ =& \{3.5, 0.4-0.7j, 0.5, 0.4+0.7j\} \end{aligned} \tag{7}$$