1. (a) For system H_1, we have:

$$y[n] = x[n] - 0.7x[n-1] - 0.6x[n-2] \tag{1}$$
$$\Rightarrow Y(z) = (1 - 0.7z^{-1} - 0.6z^{-2})X(z) \tag{2}$$
$$\Rightarrow H_1(z) = Y(z)/X(z) = (1 - 1.2z^{-1})(1 + 0.5z^{-1}) \tag{3}$$

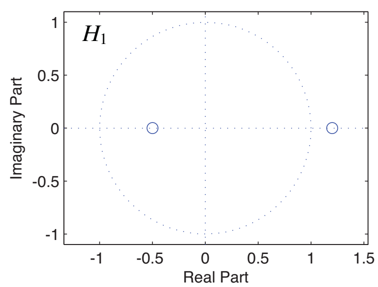
i.e. a system with zeros at $z = -0.5, 1.2$. Since it has no poles, the ROC is the entire z-plane.
For H_2, we have:

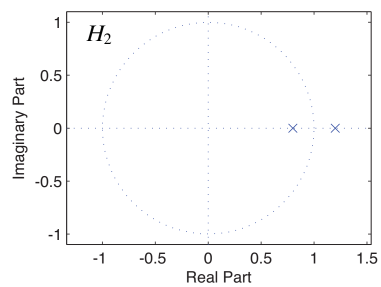
$$y[n] = x[n] + 2.0y[n-1] - 0.96y[n-2] \tag{4}$$
$$\Rightarrow Y(z) = X(z) + 2.0z^{-1}Y(z) - 0.96z^{-2})Y(z) \tag{5}$$
$$\Rightarrow Y(z)(1 - 2.0z^{-1} + 0.96z^{-2}) = X(z) \tag{6}$$
$$\Rightarrow H_2(z) = Y(z)/X(z) = \frac{1}{(1 - 1.2z^{-1})(1 - 0.8z^{-1})} \tag{7}$$

i.e. a system with poles at $z = 0.8, 1.2$. Since it is a causal system, the ROC is the exterior of the largest pole radius, i.e. $|z| > 1.2$.
Immediately we notice that system H_2 has an unstable pole i.e. a pole outside the unit circle, and that system H_1 has a zero at the same value of z (although since it's a zero, it does not imply any stability issues with H_1).

(b) Plotting the pole-zero diagrams is just placing the roots mentioned above on the z-plane:

```
>> subplot(221)
>> zplane([1 -0.7 -0.6],1)
>> subplot(222)
>> zplane(1,[1 -2 .96])
```





(c) Both systems H_3 and H_4 consist of the sequential application of H_1 and H_2, the only difference being the order in which they are applied. According to what we have looked at in class, convolution is commutative, so it should make no difference in what order they are applied (but see part (e)). In both cases, the system function is the product of the two component systems i.e.

$$H_3(z) = H_1(z)H_2(z) \tag{8}$$
$$= (1 - 1.2z^{-1})(1 + 0.5z^{-1}) \cdot \frac{1}{(1 - 1.2z^{-1})(1 - 0.8z^{-1})} \tag{9}$$
$$= \frac{1 + 0.5z^{-1}}{1 - 0.8z^{-1}} \tag{10}$$

.. where we cancel the common root at $z = 1.2$ in numerator and denominator ("pole-zero cancellation"). To find the impulse response, we can simply find the inverse z transform of this system function i.e.

$$h_3[n] = h_a[n] + 0.5h_a[n-1] \tag{11}$$

where

$$h_a[n] = IZT\left\{\frac{1}{1-0.8z^{-1}}\right\} = 0.8^n\mu[n] \tag{12}$$

so

$$\begin{aligned}
h_3[n] &= 0.8^n\mu[n] + 0.5\cdot 0.8^{n-1}\mu[n-1] && (13)\\
&= 1.625\cdot 0.8^n\mu[n] - 0.625\delta[n] \qquad \text{if we combine the two } 0.8^n \text{ terms together} && (14)\\
&= \{\underline{1}, 1.3, 1.04, 0.832, 0.6656, \ldots\} && (15)
\end{aligned}$$

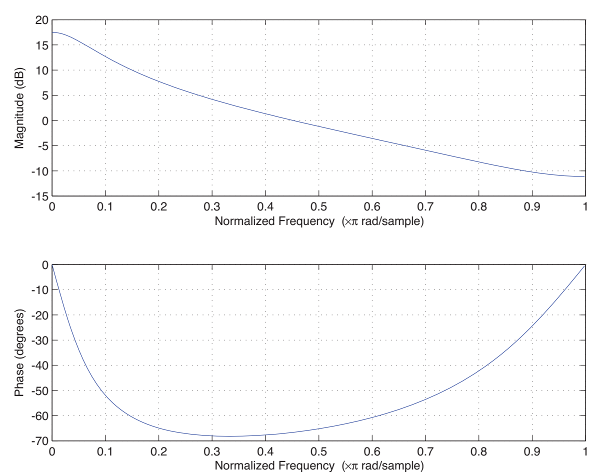
To check this, we can "run the machine" for the first few steps after inputting an impulse. From the z transform in eqn. 10, the difference equation, after canceling the coincident roots, is:

$$\begin{aligned}
y[n] - 0.8y[n-1] &= x[n] + 0.5x[n-1] && (16)\\
\Rightarrow y[n] &= x[n] + 0.5x[n-1] + 0.8y[n-1] && (17)
\end{aligned}$$

Thus, at $n = 0$ we have $y[0] = x[0] = 1$,
at $n = 1$ we have $y[1] = 0 + 0.5 + 0.8 = 1.3$,
at $n = 2$ we have $y[2] = 0 + 0 + 0.8\cdot 1.3 = 1.04$,
at $n = 3$ we have $y[3] = 0 + 0 + 0.8\cdot 1.04 = 0.832$,
... and from then on it's looking pretty good.

(d) Assuming we can cancel the pole and zero at $z = 1.2$, we have a single pole at $z = 0.8$ and a single zero at $z = -0.5$, so we expect a fairly gentle low-pass response. We can see the gain at $\omega = 0$, $|H_3(1)| = 1.5/.2 = 7.5 \approx 17.5$ dB, and at $\omega = \pi$, $|H_3(-1)| = .5/1.8 \approx 0.278 \approx -11.1$ dB. Gain will be unity (0 dB) for the point on the unit circle equidistant from both roots, which will occur on the intersection of the unit circle with the vertical line midway between the roots at $\mathbb{R}\mathrm{e}(z) = 0.15$, so $\omega = cos^{-1}(.15) \approx 0.45\pi$. That should give us enough to do our sketch, which Matlab shows as:

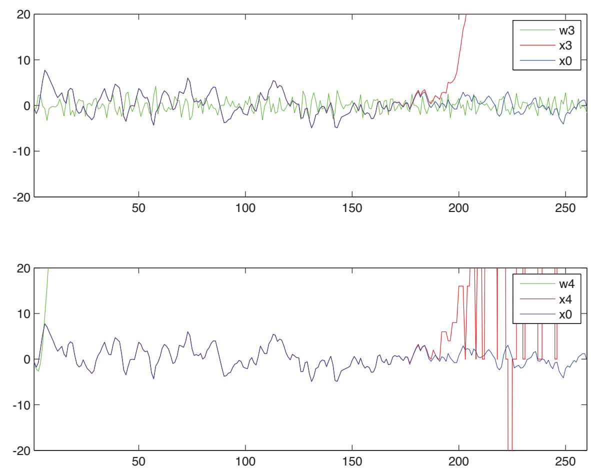
```
>> freqz([1 .5],[1 -.8])
```



(e) This is where we mention that the unstable pole in H_2 will likely cause problems even though the theoretical analysis of the entire system hides it. In particular, for H_4, the system with H_2 (with the unstable pole) as the first system in the chain, the intermediate node will be the result of applying the unstable system to an arbitrary input, and is bound to have values that "explode" (grow until it hits the representational limits used in the system.

In fact, when we attempt to simulate both versions in Matlab, we find that both versions end up diverging, although thanks to Matlab's very forgiving representation there is no particular way to predict which will last longer before giving up. In the simulation below, we plot both the outputs (`x3` and `x4`) and the intermediate node values (`w3` and `w4`) for systems fed with the same random input sequence. We see that `w4` immediately becomes very large, but the overall outputs remain about the same for the first 200 samples or so, after which the H_3 version goes unstable and diverges. A little while later, system H_4 goes unstable, by which time the intermediate value has exceeded 10^{16}.

```
>> n = randn(1,1000);
>> w3 = filter([1 -.7 -.6],1,n);
>> x3 = filter(1,[1 -2 .96],w3);
>> w4 = filter(1,[1 -2 .96],n);
>> x4 = filter([1 -.7 -.6],1,w4);
>> % ideal output
>> x0 = filter([1 .5],[1 -.8],n);
>> subplot(211)
>> plot(1:260,w3(1:260),'g', ...
1:260,x3(1:260),'r', ...
1:260,x0(1:260),'b')
>> legend('w3','x3','x0')
>> axis([1 260 -20 20])
>> subplot(212)
>> plot(1:260,w4(1:260),'g', ...
1:260,x4(1:260),'r', ...
1:260,x0(1:260),'b')
>> legend('w4','x4','x0')
>> axis([1 260 -20 20])
```



2. (a) The continuous-time signal is just a pulse of width $0.2t_p$ centered at each integer multiple of t_p. Sampling at every integer multiple of $t_p/4$ yields a periodic sequence $x_1[n] = \{\underline{1}, 0, 0, 0, 1, 0, 0, 0, \ldots\}$.

(b) A periodic, infinite sequence will have a DTFT consisting of Dirac delta functions at the multiples of the fundamental frequency – essentially a Fourier series represented within a continuous function. But let's start by trying to apply the DTFT directly:

$$X_1(e^{jw}) = \sum_n x_1[n]e^{-j\omega n} \tag{18}$$

$$= \sum_{n=r\cdot 4} e^{-j\omega n} \tag{19}$$

$$= \sum_{n'} e^{-j4\omega n'} \tag{20}$$

This won't converge because the magnitude of $e^{-j4\omega n'}$ doesn't converge, and anyway we're summing over $n' = -\infty \ldots \infty$. But we can get something that converges by averaging over an arbitrary (but long) subrange i.e. consider

$$\frac{1}{M}\sum_{n=0}^{M-1} e^{-j4\omega n} \to \begin{cases} 1 & \text{when } e^{-j4\omega} = 1 \text{ i.e. } \omega = \frac{\pi}{2}r \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Thus, we expect our DTFT to look something like:

$$X_1(e^{jw}) = K \sum_{r=0,\pm1,\pm2,\ldots} \delta(\omega - \frac{\pi}{2}r) \tag{22}$$

Notice that the 2π periodicity of $X(e^{j\omega})$ is present here. To get the actual weights of the Dirac delta functions, we can look to the IDTFT:

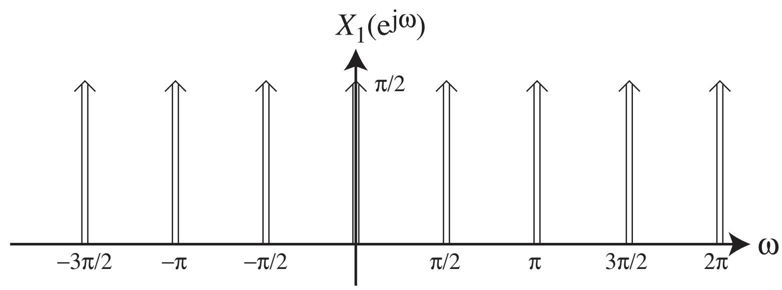
$$x_1[n] = \frac{1}{2\pi}\int_{-\pi+\epsilon}^{\pi+\epsilon} X_1(e^{j\omega})e^{j\omega n}d\omega \tag{23}$$

$$= \frac{1}{2\pi}K\sum_{r=-1}^{2} e^{j\frac{\pi}{2}rn} \tag{24}$$

For $n = 0$, where we know $x_1 = 1$, we get $x_1[0] = \frac{K}{2\pi}4$, so $K = \pi/2$, and thus:

$$X_1(e^{jw}) = \frac{\pi}{2}\sum_r \delta(\omega - \frac{\pi}{2}r) \tag{25}$$

We sketch this with little arrows to indicate the delta functions:



(c) $x_2[n]$ is simply $\{\underline{1}, 0, 0, 0, 1, 0, 0, 0\}$ whose DFT we can get directly from the DFT equation:

$$X_2[k] = \sum_{n=0}^{N} x[n]e^{-j\frac{2\pi kn}{N}} \tag{26}$$

$$= 1 + e^{-j2\pi 4k/8} = 1 + e^{-j\pi k} \tag{27}$$

$$= 1 + (-1)^k = \{\underline{2}, 0, 2, 0, 2, 0, 2, 0\} \tag{28}$$

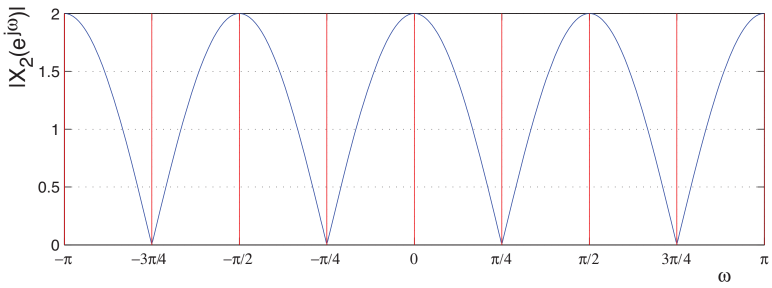
(d) The DTFT is pretty much the same, but with ω taking the place of $2\pi k/N$:

$$X_2(e^{j\omega}) = \sum_n x[n]e^{-j\omega n} \tag{29}$$

$$= 1 + e^{-j4\omega} \tag{30}$$

$$= e^{-j2\omega}(e^{j2\omega} + e^{-j2\omega}) \tag{31}$$

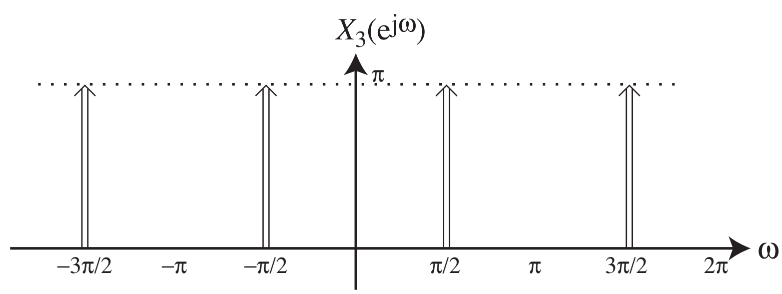
$$= 2e^{-j2\omega}\cos(2\omega) \tag{32}$$



As shown in the sketch, when we sample the 2π cycle of $H_2(e^{j\omega})$ at N equally-spaced intervals (for $N = 8$), we get exactly the sequence found in part (c), as expected. (The $e^{-2j\omega}$ and the sign flips in the $\cos(2\omega)$ take care of another).

The relationship to part (b) is more complex, but one way to explain it is that we have windowed $x_1[n]$ in the time domain with a rectangular window of 8 points, which leads to convolving their responses in the Fourier domain. Since $X_1(e^{j\omega})$ is simply a series of Dirac deltas, the convolution results in placing shifted versions of the Fourier transform of the rectangular window (a sinc function whose main lobe extends from $-\pi/4$ to $\pi/4$) centered on each nonzero value in X_1. This explains the peaks, although it is less obvious that the shape of these superimposed pulses ends up as exactly a cosine.

(e) We notice that $H_3(e^{j\omega})$ is the DTFT of $\{\underline{1}, 0, -1\}$. Thus, the convolution consists of adding the original sequence to a sign-flipped, delayed-by-two version, or $x_3[n] = \{\underline{1}, 0, -1, 0, 1, 0, -1, 0, \ldots\}$. We also notice that this is precisely the sampled version of a cosine which completes one cycle every four samples i.e. $\cos(n\pi/2)$, so the DTFT must be $X_3(e^{j\omega}) = \pi \sum_r \delta(\omega - \pi/2 + 2\pi r) + \delta(\omega + \pi/2 + 2\pi r)$, and the sketch is as shown below:



This is consistent, of course, with multiplying the DTFTs for X_1 and H_1, since X_1 is only nonzero at $\omega = r \cdot \pi/2$, and H_1 is evidently 0 for even r, and 2 for odd r, leading to the pruned and scaled set of Dirac deltas in X_3.