EE E6820: Speech & Audio Processing & Recognition

Lecture 3:
Acoustics

1. The wave equation
2. Acoustic tubes: reflections & resonance
3. Oscillations & musical acoustics
4. Spherical waves & room acoustics

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Acoustics & sound

- Acoustics is the study of physical waves
- (Acoustic) waves transmit energy without permanently displacing matter (e.g. ocean waves)
- Same math recurs in many domains
- Intuition: pulse going down a rope
The wave equation

- Consider a small section of the rope:

- displacement is $y(x)$, tension $S$, mass $\varepsilon \cdot dx$

  $\rightarrow$ lateral force is
  
  $$F_y = S \cdot \sin(\phi_2) - S \cdot \sin(\phi_1)$$
  
  $$= S \cdot \frac{\partial^2 y}{\partial x^2} \cdot dx$$
Wave equation (2)

• Newton’s law: \( F = ma \)

\[
S \cdot \frac{\partial^2 y}{\partial x^2} \cdot dx = \varepsilon dx \cdot \frac{\partial^2 y}{\partial t^2}
\]

• Call \( c^2 = S/\varepsilon \) (tension to mass-per-length)

hence the wave equation:

\[
c^2 \cdot \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}
\]

.. partial DE relating curvature and acceleration
Solution to the wave equation

- If \( y(x, t) = f(x - ct) \)

then

\[
\frac{\partial y}{\partial x} = f'(x - ct) \quad \frac{\partial y}{\partial t} = -c \cdot f'(x - ct)
\]

\[
\frac{\partial^2 y}{\partial x^2} = f''(x - ct) \quad \frac{\partial^2 y}{\partial t^2} = c^2 \cdot f''(x - ct)
\]

also works for \( y(x, t) = f(x + ct) \)

Hence, general solution:

\[
c^2 \cdot \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}
\]

\[
\Rightarrow \quad y(x, t) = y^+(x - ct) + y^-(x + ct)
\]
Solution to the wave equation (2)

- $y^+(x - ct)$ and $y^-(x + ct)$ are travelling waves
- shape stays constant but changes position:

\[
y^+(x - ct) \quad \text{and} \quad y^-(x + ct) \quad \text{are travelling waves}
\]

- shape stays constant but changes position:

\[
y^+(x - ct) \quad \text{and} \quad y^-(x + ct)
\]

- resultant $y(x)$ is \textit{sum} of the two waves

\[
\Delta x = c \cdot T
\]

- $c$ is travelling wave velocity ($\Delta x / \Delta t$)

- $y^+$ moves right, $y^-$ moves left

\[
y^+(x - ct) \quad \text{and} \quad y^-(x + ct)
\]

- resultant $y(x)$ is \textit{sum} of the two waves
Wave equation solutions (3)

- **What is the form of \( y^+, y^- \)?**
  - any doubly-differentiable function will satisfy wave equation

- **Actual waveshapes dictated by boundary conditions**
  - \( y(x) \) at \( t = 0 \)
  - constraints on \( y \) at particular \( x \)'s
    - e.g. input motion \( y(0, t) = m(t) \)
    - rigid termination \( y(L, t) = 0 \)

\[
y(0,t) = m(t) \\
y^+(x,t) \\
y(L,t) = 0
\]
Terminations and reflections

- **System constraints:**
  - initial \( y(x, 0) = 0 \) (flat rope)
  - input \( y(0, t) = m(t) \) (at agent’s hand) \( \rightarrow y^+ \)
  - termination \( y(L, t) = 0 \) (fixed end)
  - wave equation \( y(x,t) = y^+(x - ct) + y^-(x + ct) \)

- **At termination:**
  \[ y(L, t) = 0 \rightarrow y^+(L - ct) = - y^-(L + ct) \]
  i.e. \( y^+ \) and \( y^- \) are mirrored in *time* and *amplitude*
  around \( x=L \)
  \( \rightarrow \) inverted reflection at termination

\[ \rightarrow \text{simulation [travel1.m]} \]
Acoustic tubes

- Sound waves travel down acoustic tubes:
  - 1-dimensional; very similar to strings

- Common situation:
  - wind instrument bores
  - ear canal
  - vocal tract
Pressure and velocity

- Consider air particle displacement $\xi(x, t)$:

\[
\frac{\partial}{\partial t} \xi(x, t) = ut \quad \text{and} \quad \frac{\partial}{\partial x} \xi(x, t) = -\frac{1}{\kappa} \frac{\partial p}{\partial x}
\]

- Particle velocity $v(x, t) = \frac{\partial \xi}{\partial t}$

hence volume velocity $u(x, t) = A \cdot v(x, t)$

- Air pressure $p(x, t) = -\frac{1}{\kappa} \frac{\partial \xi}{\partial x}$
Wave equation for a tube

- **Consider elemental volume:**
  - Area $dA$
  - Force $p \cdot dA$
  - Volume $dA \cdot dx$
  - Force $(p + \frac{\partial p}{\partial x} \cdot dx) \cdot dA$
  - Mass $\rho \cdot dA \cdot dx$

- **Newton's law:** $F = ma$

\[-\frac{\partial p}{\partial x} \cdot dx \cdot dA = \rho dA dx \cdot \frac{\partial v}{\partial t}\]

\[\Rightarrow \frac{\partial p}{\partial x} = -\rho \frac{\partial v}{\partial t}\]

- **Hence**

\[c^2 \cdot \frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2}\]

\[c = \frac{1}{\sqrt{\rho \kappa}}\]
Acoustic tube traveling waves

- Traveling waves in particle displacement:
  \[ \xi(x, t) = \xi^+(x - ct) + \xi^-(x + ct) \]

- Call \( u^+(\alpha) = -cA \frac{\partial}{\partial \alpha} \xi^+(\alpha) \)

\[ Z_0 = \frac{\rho c}{A} \]

- Then pressure:
  \[ p(x, t) = -\frac{1}{\kappa} \cdot \frac{\partial \xi}{\partial x} = Z_0 \cdot [u^+(x - ct) + u^-(x + ct)] \]

- Volume velocity:
  \[ u(x, t) = A \cdot \frac{\partial \xi}{\partial t} = u^+(x - ct) - u^-(x + ct) \]

- (Scaled) sum and difference of traveling waves
Acoustic tube traveling waves (2)

- Different residuals for pressure and vol. veloc.:

\[ u^+ - u^- \]
\[ p(x,t) = Z_0[u^+ + u^-] \]
Terminations in tubes

- **Equivalent of ‘fixed point’ for tubes?**

  Solid wall forces
  \[ u(x,t) = 0 \quad \text{hence} \quad u^+ = u^- \]

  Open end forces
  \[ p(x,t) = 0 \]
  \[ \text{hence} \quad u^+ = -u^- \]

- **Open end** is like fixed point for rope:
  reflects wave back inverted

- **Unlike fixed point, solid wall reflects traveling wave without inversion**
Standing waves

- Consider (complex) sinusoidal input:
  \[ u_0(t) = U_0 \cdot e^{j\omega t} \]

- At any point, values will have form \( Ke^{j(\omega t + \phi)} \)

- Hence traveling waves:
  \[ u^+(x - ct) = \left| A \right| e^{j(-kx + \omega t + \phi_A)} \]
  \[ u^-(x + ct) = \left| B \right| e^{j(kx + \omega t + \phi_B)} \]

where \( k = \frac{\omega}{c} \) (spatial frequency, rad/m)
(wavelength \( \lambda = \frac{c}{f} = \frac{2\pi c}{\omega} \))

- Pressure / vol. veloc. resultants show stationary pattern: standing waves
- even when \( |A| \neq |B| \)
  \( \rightarrow \) simulation [sintwavemov.m]
Standing waves (2)

For lossless termination ($|\vec{u}^+| = |\vec{u}^-|$), have true nodes & antinodes

Pressure and vol. veloc. are phase shifted
- in space and in time

*
Transfer function

- Consider tube excited by $u_0(t) = U_0 \cdot e^{j\omega t}$:
  - sinusoidal traveling waves must satisfy termination ‘boundary conditions’
  - satisfied by complex constants $A$ and $B$ in
  
  $u(x, t) = u^+(x - ct) + u^-(x + ct)$
  
  $= Ae^{j(-kx + \omega t)} + Be^{j(kx + \omega t)}$

  $= e^{j\omega t} \cdot (A e^{-jkx} + Be^{jkx})$

  - standing wave pattern will scale with input magnitude
  - point of excitation makes a big difference
Transfer function (2)

- For open-ended tube of length $L$ excited at $x = 0$ by $U_0 e^{j\omega t}$:
  \[
  u(x, t) = \frac{\cos k(L - x)}{\cos kL} \cdot U_0 e^{j\omega t} \quad \left(k = \frac{\omega}{c}\right)
  \]
  (works at $x = 0$)

- i.e. standing wave pattern
  e.g. varying $L$ for a given $\omega$ (and hence $k$):

![Diagram of standing wave pattern](image)
Transfer function (3)

- **Varying** $\omega$ **for a given** $L$:

  - at $x = L$, \[ \frac{u(L, t)}{u(0, t)} = \frac{1}{\cos kL} = \frac{1}{\cos (\omega L/c)} \]

  \[ \frac{u(L)}{u(0)} \rightarrow \infty \quad \text{at} \quad \omega L/c = (2r+1)\pi/2, \; r = 0,1,2,... \]

  - **Output vol. veloc. always larger than input**

  - **Unbounded for** $L = (2r + 1) \frac{\pi c}{2\omega} = (2r + 1) \frac{\lambda}{4}$

    i.e. resonance
Resonant modes

- For lossless tube

\[ L = m \cdot \frac{\lambda}{4}, \quad m \text{ odd, } \lambda \text{ wavelength,} \]

\[ \left| \frac{u(L)}{u(0)} \right| \text{ is unbounded, meaning:} \]

- transfer function has pole on frequency axis
- energy at that frequency sustains indefinitely

\[ L = 3 \cdot \lambda_{1/4} \]

\[ \omega_1 = 3\omega_0 \]

- e.g 17.5 cm vocal tract, \( c = 350 \text{ m/s} \)

\[ \omega_0 = 2\pi \cdot 500 \text{ Hz (then 1500, 2500 ...)} \]
Scattering junctions

At abrupt change in area:
- pressure must be continuous
  \[ p_k(x, t) = p_{k+1}(x, t) \]
- vol. veloc. must be continuous
  \[ u_k(x, t) = u_{k+1}(x, t) \]
- traveling waves
  \( u^+_k, u^-_k, u^+_{k+1}, u^-_{k+1} \)
  will be different

**Solve e.g. for \( u^-_k \) and \( u^+_{k+1} \):** (generalized term.)

\[
r = \frac{A_{k+1}}{A_k}
\]

"Area ratio"
Concatenated tube model

• Discrete approximation to varying-diameter tube:

\[ A_k, L_k \rightarrow A_{k+1}, L_{k+1} \]

• Can solve for resonances
• Reasonable approx to human vocal tract
• Vowel formants from tube resonances

sound example? ah ee oo
3 Oscillations & musical acoustics

• Pitch (periodicity) is essence of music:

- why? why music?

• Different kinds of oscillators:
  - simple harmonic motion (tuning fork)
  - relaxation oscillator (voice)
  - string traveling wave (plucked/struck/bowed)
  - air column (nonlinear energy element)
Simple harmonic motion

• Basic mechanical oscillation:

\[ \ddot{x} = -\omega^2 x \quad x = A \cos(\omega t + \varphi) \]

• Spring + mass (+ damper)

\[ F = kx \]

\[ \omega^2 = \frac{k}{m} \]

• e.g. tuning fork

• Not great for music:
  - fundamental (cos\(\omega t\)) only
  - relatively low energy
Relaxation oscillator

- **Multi-state process:**
  - one state builds up potential (e.g. pressure)
  - switch to second (release) state
  - revert to first state etc.

- **e.g. vocal folds:**

- **Oscillation period depends on force (tension)**
  - easy to change
  - hard to keep stable
  → less used in music
Ringing string

- e.g. our original ‘rope’ example
  \[ \omega^2 = \frac{\pi^2 S}{L^2 \varepsilon} \]

  \[ \text{mass/length } \varepsilon \]
  \[ \text{tension } S \]

- Many musical instruments
  - guitar (plucked)
  - piano (struck)
  - violin (bowed)

- Control period (pitch):
  - change length (fretting)
  - change tension (tuning piano)
  - change mass (piano strings)

- Influence of excitation ... [pluck1a.m]
Wind tube

- **Resonant tube + energy input**

- e.g. clarinet
  - lip pressure keeps reed closed
  - reflected pressure wave opens reed
  - reinforced pressure wave passes through

- **Finger holes determine first reflection**
  → effective waveguide length

\[ \omega = \frac{\pi c}{2L} \text{ (quarter wavelength)} \]
Room acoustics

- Sound in free air expands spherically:

\[
\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial p}{\partial r} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}
\]

solved by

\[
p(r, t) = \frac{P_0}{r} e^{j(\omega t - kr)}
\]

- Intensity \( \propto p^2 \) falls as \( \frac{1}{r^2} \)
Effect of rooms (1): Images

- Ideal reflections are like multiple sources:
  - Virtual (image) sources
  - Reflected path
  - Direct path
  - Early echos in room impulse response:
  - Actual reflection may be $h_{ref}(t)$, not $\delta(t)$
Effect of rooms (2): modes

- Regularly-spaced echos behave like acoustic tubes:

- Real rooms have lots of modes!
  - dense, sustained echos in impulse response
  - complex pattern of peaks in frequency response
Reverberation

- Exponential decay of reflections:
  \[ h_{\text{room}}(t) \sim e^{-t/\tau} \]

- Frequency-dependent
  - greater absorption at high frequencies → faster decay

- Size-dependent
  - larger rooms → longer delays → slower decay

- Sabine’s equation:
  \[ RT_{60} = \frac{0.049 V}{S \bar{\alpha}} \]

- Time constant as size, absorption [e.g.]
Summary

• Travelling waves
• Acoustic tubes & resonance
• Musical acoustics & periodicity
• Room acoustics & reverberation