1. Filter Design Specifications

- The filter design process:
  - Problem
  - Analysis
    - performance constraints
    - magnitude response
    - phase response
    - cost/complexity
  - Design
    - $G(z)$ transfer function
    - FIR/IIR
    - subtype
    - order
  - Implement
    - platform
    - structure
    - ...
  - Solution
Performance Constraints

- .. in terms of magnitude response:

![Graph showing magnitude response with labeled performance constraints]

- "Best" filter:
  - Narrowest Transition Band
  - Smallest Passband Ripple
  - Greatest Minimum SB Attenuation

- Improving one usually worsens others
- But: increasing filter order (i.e. cost) improves all three measures
Passband Ripple

- Assume peak passband gain = 1 then minimum passband gain = \[
\frac{1}{\sqrt{1 + \epsilon^2}}
\]
- Or, ripple \( \alpha_{\text{max}} = 20 \log_{10} \sqrt{1 + \epsilon^2} \) dB

Stopband Ripple

- Peak passband gain is \( A \times \) larger than peak stopband gain
- Hence, minimum stopband attenuation \( \alpha_s = -20 \log_{10} \frac{1}{A} = 20 \log_{10} A \) dB
### Filter Type Choice: FIR vs. IIR

<table>
<thead>
<tr>
<th>FIR</th>
<th>IIR</th>
</tr>
</thead>
<tbody>
<tr>
<td>- No feedback (just zeros)</td>
<td>- Feedback (poles &amp; zeros)</td>
</tr>
<tr>
<td>- Always stable</td>
<td>- May be unstable</td>
</tr>
<tr>
<td>- Can be linear phase</td>
<td>- Difficult to control phase</td>
</tr>
<tr>
<td><strong>BUT</strong> - High order (20-2000)</td>
<td>- Typ. &lt; 1/10th order of FIR (4-20)</td>
</tr>
<tr>
<td>- Unrelated to continuous-time filtering</td>
<td>- Derive from analog prototype</td>
</tr>
</tbody>
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### FIR vs. IIR

- If you care about **computational cost** → use low-complexity **IIR**
  (computation no object → Lin Phs FIR)
- If you care about **phase response** → use linear-phase **FIR**
  (phase unimportant → go with simple IIR)
IIR Filter Design

- IIR filters are directly related to analog filters (continuous time)
  - via a mapping of $H(s)$ (CT) to $H(z)$ (DT) that preserves many properties
- Analog filter design is sophisticated
  - signal processing research since 1940s
  - Design IIR filters via analog prototype
  - hence, need to learn some CT filter design

2. Analog Filter Design

- Decades of analysis of transistor-based filters – sophisticated, well understood
- Basic choices:
  - ripples vs. flatness in stop and/or passband
  - more ripples $\rightarrow$ narrower transition band

<table>
<thead>
<tr>
<th>Family</th>
<th>PB</th>
<th>SB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Butterworth</td>
<td>flat</td>
<td>flat</td>
</tr>
<tr>
<td>Chebyshev I</td>
<td>ripples</td>
<td>flat</td>
</tr>
<tr>
<td>Chebyshev II</td>
<td>flat</td>
<td>ripples</td>
</tr>
<tr>
<td>Elliptical</td>
<td>ripples</td>
<td>ripples</td>
</tr>
</tbody>
</table>
CT Transfer Functions

- Analog systems: $s$-transform (Laplace)

**Continuous-time**

$$H_a(s) = \int h_a(t)e^{-st}dt$$

$$H_a(j\Omega)$$

**Discrete-time**

$$H_d(z) = \sum h_d[n]z^{-n}$$

$$H_d(e^{j\omega})$$

<table>
<thead>
<tr>
<th>Pole/zero diagram</th>
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<tr>
<td>Continuous-time</td>
</tr>
<tr>
<td>Discrete-time</td>
</tr>
</tbody>
</table>

- Transform
- Frequency response
- Pole/zero diagram

**Butterworth Filters**

**Maximally flat in pass and stop bands**

- Magnitude response (LP):

  $$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

  - $\Omega \ll \Omega_c$, $|H_a(j\Omega)|^2 \rightarrow 1$
  - $\Omega = \Omega_c$, $|H_a(j\Omega)|^2 = \frac{1}{2}$

  3dB point

  ![Butterworth Filter Diagram]
Butterworth Filters

- $\Omega \gg \Omega_c$, $\left| H_a(j\Omega) \right|^2 \rightarrow (\Omega_c/\Omega)^{2N}$

Log-log magnitude response

$\text{flat} \rightarrow \frac{d^n}{d\Omega^n} \left| H_a(j\Omega) \right|^2 = 0$

$\Omega = 0$ for $n = 1 \ldots 2N-1$

6N dB/oct rolloff

Butterworth Filters

- How to meet design specifications?

$$\frac{1}{\sqrt{1 + \varepsilon^2}} \left| H_a(j\Omega) \right| = \frac{1}{1 + \left( \frac{\Omega_p}{\Omega} \right)^{2N}} = \frac{1}{1 + \varepsilon^2}$$

Design Equation

$$N \geq \frac{1}{2} \log_{10} \left( \frac{\Omega_p}{\Omega_s} \right)$$

- $k_1 = \frac{\varepsilon}{\sqrt{A^2 - 1}}$ = “discrimination”, << 1
- $k = \frac{\Omega_p}{\Omega_s}$ = “selectivity”, < 1
Butterworth Filters

- \[ |H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} \]
  but what is \( H_a(s) \)?

- Traditionally, look it up in a table
  - calculate \( N \) \( \rightarrow \) normalized filter with \( \Omega_c = 1 \)
  - scale all coefficients for desired \( \Omega_c \)

- In fact,
  \[ H_a(s) = \prod_i \left( \frac{1}{s-p_i} \right) \]
  where \( p_i = \Omega_c e^{j\pi N + \frac{2i-1}{2N}} i = 1..N \)

Butterworth Example

Design a Butterworth filter with 1 dB cutoff at 1 kHz and a minimum attenuation of 40 dB at 5 kHz

\[-1\text{dB} = 20\log_{10} \frac{1}{\sqrt{1 + \varepsilon^2}} \Rightarrow \varepsilon^2 = 0.259\]

\[-40\text{dB} = 20\log_{10} \frac{1}{A} \Rightarrow A = 100\]

\[ \frac{\Omega_s}{\Omega_p} = 5 \]

\[ N \geq \frac{1}{2} \log_{10} \frac{9999}{0.259} \]

\[ \Rightarrow N = 4 \geq 3.28 \]
Butterworth Example

- Order \( N = 4 \) will satisfy constraints; What are \( \Omega_c \) and filter coefficients?
  - from a table, \( \Omega_{-1dB} = 0.845 \) when \( \Omega_c = 1 \)  
    \[ \Rightarrow \Omega_c = \frac{1000}{0.845} = 1.184 \text{ kHz} \]
  - from a table, get normalized coefficients for \( N = 4 \), scale by 1184
  - Or, use Matlab:
    \[ [b,a] = \text{butter}(N,Wc,'s'); \]

Chebyshev I Filter

- Equiripple in passband (flat in stopband)  
  \( \rightarrow \) minimize maximum error

\[
|H_a(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 T_N^2 \left( \frac{\Omega}{\Omega_p} \right)}
\]

\( T_N(\Omega) = \begin{cases} 
\cos(N \cos^{-1} \Omega) & |\Omega| \leq 1 \\
\cosh(N \cosh^{-1} \Omega) & |\Omega| > 1
\end{cases} \)
Chebyshev I Filter

■ Design procedure:
  ■ desired passband ripple $\rightarrow \varepsilon$
  ■ min. stopband atten., $\Omega_p, \Omega_s \rightarrow N$

\[
\frac{1}{A^2} = \frac{1}{1 + \varepsilon^2 T_N^2 \left( \frac{\Omega_s}{\Omega_p} \right)} = \frac{1}{1 + \varepsilon^2 \left[ \cosh \left( N \cosh^{-1} \frac{\Omega_s}{\Omega_p} \right) \right]^2}
\]

$\Rightarrow N \geq \frac{\cosh^{-1} \left( \sqrt{\frac{A^2 - 1}{\varepsilon}} \right)}{\cosh^{-1} \left( \frac{\Omega_s}{\Omega_p} \right)}$

$\frac{1}{k_1}$, discrimination

$\frac{1}{k}$, selectivity

Chebyshev I Filter

■ What is $H_a(s)$?
  ■ complicated, get from a table
  ■ .. or from Matlab cheby1($N, r, Wp, 's'$)
  ■ all-pole; can inspect them:

..like squashed-in Butterworth
Chebyshev II Filter

- Flat in passband, equiripple in stopband

\[ |H_d(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \left( \frac{T_N(\Omega_s)}{T_N(\Omega_p)} \right)^2} \]

- Filter has poles and zeros (some)
- Complicated pole/zero pattern

Elliptical (Cauer) Filters

- Ripples in both passband and stopband

\[ |H_d(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 R_N^2(\Omega/\Omega_p)} \]

- Complicated; not even closed form for \( N \)
Analog Filter Types Summary

Butterworth

Chebyshev I

Chebyshev II

Elliptical

\[ \Omega_C \]

\[ \Omega_p \]

\[ N = 6 \]

\[ r = 3 \text{ dB} \]

\[ A = 40 \text{ dB} \]

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Analog Filter Transformations

- All filters types shown as lowpass; other types (highpass, bandpass..) derived via transformations
  - i.e. \[ \hat{s} = F^{-1}(s) \]
  - Lowpass prototype \( H_{LP}(s) \) \[ \rightarrow \] Desired alternate response; still a rational polynomial
  - General mapping of \( s \)-plane
    - \( BUT \) keep \( j\Omega \) \( \rightarrow \) \( j\hat{\Omega} \);
    - frequency response just ‘shuffled’

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**Lowpass-to-Highpass**

- Example transformation:
  \[ H_{HP}(\hat{s}) = H_{LP}(s)|_{s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}}} \]
  - take prototype \( H_{LP}(s) \) polynomial
  - replace \( s \) with \( \frac{\Omega_p \hat{\Omega}_p}{\hat{s}} \)
  - simplify and rearrange
    \[ \rightarrow \text{new polynomial } H_{HP}(\hat{s}) \]

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**Lowpass-to-Highpass**

- What happens to frequency response?
  \[ s = j\Omega \quad \Rightarrow \quad \hat{s} = \frac{\Omega_p \hat{\Omega}_p}{j\Omega} = j \left( -\frac{\Omega_p \hat{\Omega}_p}{\Omega} \right) \]
  \[ \Rightarrow \hat{\Omega} = \frac{-\Omega_p \hat{\Omega}_p}{\Omega} \quad \text{imaginary axis stays on self...} \]
  \[ \text{...freq. } \rightarrow \text{freq.} \]
  - \( \Omega = \Omega_p \quad \Rightarrow \quad \hat{\Omega} = -\hat{\Omega}_p \)
    \[ \text{LP passband} \quad \text{HP passband} \]
  - \( \Omega < \Omega_p \quad \Rightarrow \quad \hat{\Omega} < -\hat{\Omega}_p \)
  - \( \Omega > \Omega_p \quad \Rightarrow \quad \hat{\Omega} > -\hat{\Omega}_p \)
    \[ \text{LP stopband} \quad \text{HP stopband} \]
  - Frequency axes inverted
Transformation Example

Design a Butterworth highpass filter with PB edge -0.1dB @ 4 kHz ($\hat{\Omega}_p$) and SB edge -40 dB @ 1 kHz ($\hat{\Omega}_s$)

- Lowpass prototype: make $\Omega_p = 1$
  \[ \Rightarrow \Omega_s = \left( -\frac{\hat{\Omega}_p}{\hat{\Omega}_s} \right) = (-)4 \]

- Butterworth -0.1dB @ $\Omega_p = 1$, -40dB @ $\Omega_s = 4$

\[ N \geq \frac{1}{2} \log_{10} \left( \frac{\frac{\hat{\Lambda}^2-1}{\hat{\epsilon}^2}}{\frac{\hat{\Omega}_s}{\hat{\Omega}_p}} \right) \]
\[ \Omega_p @ -0.1dB \Rightarrow \frac{1}{1 + \left( \frac{\hat{\Omega}_p}{\hat{\Omega}_c} \right)^{10}} = 10^{-0.1} \]
\[ \Rightarrow N = 5 \quad \Rightarrow \Omega_c = \frac{\Omega_p}{0.6866} = 1.4564 \]

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Transformation Example

- LPF proto has $p_{\ell} = \Omega_c e^{j\pi N + \ell - 1} 2N$
  \[ \Rightarrow H_{LP}(s) = \frac{\Omega_c^N}{\prod_{i=1}^{N} (s - p_{i\ell})} \]

- Map to HPF: $H_{HP}(\hat{s}) = H_{LP}(s) \bigg|_{s = \frac{\hat{\Omega}_p \hat{\Omega}_p}{\hat{s}}}^N$

\[ \Rightarrow H_{HP}(\hat{s}) = \prod_{\ell=1}^{N} \left( \frac{\Omega_c^N}{\hat{s}} - p_{\ell\hat{s}} \right) = \prod_{\ell=1}^{N} \left( \hat{s}^N \right) \]
\[ \Rightarrow N \text{ zeros at } \hat{s} = \Omega_{\hat{s}} = 0 \]

\[ \text{new poles at } \hat{s} = \Omega_{\hat{s}} = \frac{\hat{s}^N}{p_{i\hat{s}}} \]

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Transformation Example

- In Matlab:
  \[
  [N, Wc] = \text{buttord}(1, 4, 0.1, 40, 's');
  [B, A] = \text{butter}(N, Wc, 's');
  [n, d] = \text{lp2hp}(B, A, 2\times\pi\times4000);
  \]

3. Analog Protos → IIR Filters

- Can we map high-performance CT filters to DT domain?
- Approach: transformation \( H_a(s) \rightarrow G(z) \)
  \[
  G(z) = H_a(s)|_{s = F(z)}
  \]
  where \( s = F(z) \) maps \( s \)-plane ↔ \( z \)-plane:

  Every value of \( G(z) \) is a value of \( H_a(s) \) somewhere on the \( s \)-plane & vice-versa
CT to DT Transformation

- Desired properties for \( s = F(z) \):
  - \( s \)-plane \( j\Omega \) axis \( \leftrightarrow \) \( z \)-plane unit circle
    - preserves frequency response values
  - \( s \)-plane LHHP \( \leftrightarrow \) \( z \)-plane unit circle interior
    - preserves stability of poles

\[ s = \frac{1 - z^{-1}}{1 + z^{-1}} \quad \text{Bilinear Transform} \]

- Hence inverse:
  \[ z = \frac{1 + s}{1 - s} \]
  - unique, 1:1 mapping

- Freq. axis? \( s = j\Omega \) \( \Rightarrow z = \frac{1 + j\Omega}{1 - j\Omega} \quad |z| = 1 \) i.e. on unit circle

- Poles? \( s = \sigma + j\Omega \) \( \Rightarrow z = \frac{(1+\sigma)+j\Omega}{(1-\sigma)-j\Omega} \quad \sigma < 0 \)
  \( \Rightarrow |z|^2 = \frac{1 + 2\sigma + \sigma^2 + \Omega^2}{1 - 2\sigma + \sigma^2 + \Omega^2} \quad \Leftrightarrow |z| < 1 \)
Bilinear Transformation

- How can entire half-plane fit inside u.c.?

- Highly nonuniform warping!

Bilinear Transformation

- What is CT↔DT freq. relation Ω↔ω?

\[ z = e^{j\omega} \Rightarrow s = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{2j \sin \omega / 2}{2\cos \omega / 2} = j \tan \frac{\omega}{2} \text{ im.axis} \]

- i.e. \[ \Omega = \tan \left( \frac{\omega}{2} \right) \]
  \[ \omega = 2 \tan^{-1} \Omega \]

- *infinite* range of CT frequency \(-\infty < \Omega < \infty\) maps to *finite* DT freq. range \(-\pi < \omega < \pi\)

- nonlinear: \[ \frac{d}{d\omega} \Omega \rightarrow \infty \text{ as } \omega \rightarrow \pi \] pack it all in!
Frequency Warping

- Bilinear transform makes
  \[ G\left(e^{j\omega}\right) = H_a\left(j\Omega\right)\big|_{\omega=2\tan^{-1}\Omega} \]
  for all \( \omega, \Omega \)

  - Same gain & phase (\( \varepsilon, A \ldots \)), in same ‘order’, but with *warped* frequency axis

Design Procedure

- Obtain DT filter specs:
  - general form (LP, HP...), \( \omega_p, \omega_s, \frac{1}{\sqrt{1+\varepsilon^2}}, \frac{1}{A} \)

- ‘Warp’ frequencies to CT:
  - \( \Omega_p = \tan\frac{\omega_p}{2} \), \( \Omega_s = \tan\frac{\omega_s}{2} \)

  - Design analog filter for \( \Omega_p, \Omega_s, \frac{1}{\sqrt{1+\varepsilon^2}}, \frac{1}{A} \)
    \[ \rightarrow H_a(s), \text{ CT filter polynomial} \]

  - Convert to DT domain: \( G(z) = H_a\left(s\right)|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \)
    \[ \rightarrow G(z), \text{ rational polynomial in } z \]

  - Implement digital filter!
Bilinear Transform Example

- DT domain requirements:
  Lowpass, 1 dB ripple in PB, $\omega_p = 0.4\pi$, SB attenuation $\geq 40$ dB @ $\omega_s = 0.5\pi$, attenuation increases with frequency

- SB ripples, PB monotonic
  $\rightarrow$ Chebyshev I

Bilinear Transform Example

- Warp to CT domain:
  $\Omega_p = \tan \frac{\omega_p}{2} = \tan 0.2\pi = 0.7265$ rad/sec
  $\Omega_s = \tan \frac{\omega_s}{2} = \tan 0.25\pi = 1.0$ rad/sec

- Magnitude specs:
  1 dB PB ripple
  $\Rightarrow \frac{1}{\sqrt{1+\varepsilon^2}} = 10^{-1/20} = 0.8913 \Rightarrow \varepsilon = 0.5087$

  40 dB SB atten.
  $\Rightarrow \frac{1}{A} = 10^{-40/20} = 0.01 \Rightarrow A = 100$
**Bilinear Transform Example**

- Chebyshev I design criteria:
  \[
  N \geq \frac{\cosh^{-1}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = 7.09 \quad \text{i.e. need } N = 8
  \]

- Design analog filter, map to DT, check:

```matlab
>> N=8;
>> wp=0.7265;
>> [B,A]=cheby1(N,1,wp,'s');
>> [b,a] = bilinear(B,A,.5);
```

**Other Filter Shapes**

- Example was IIR LPF from LP prototype
- For other shapes (HPF, bandpass,...):

  ![Filter Shapes Diagram]

  - **Transform** LP→X in CT or DT domain...
DT Spectral Transformations

- Same idea as CT LPF → HPF mapping, but in z-domain:
  \[ G_D(\hat{z}) = G_L(z) \bigg|_{z=F(\hat{z})} = G_L(F(\hat{z})) \]

- To behave well, \( z = F(\hat{z}) \) should:
  - map u.c. → u.c. (preserve \( G(e^{j\omega}) \) values)
  - map u.c. interior → u.c. interior (stability)
- i.e. \( |F(\hat{z})| = 1 \Leftrightarrow |\hat{z}| = 1 \quad |F(\hat{z})| < 1 \Leftrightarrow |\hat{z}| < 1 \)
- in fact, \( F(\hat{z}) \) matches the definition of an allpass filter ... replace delays with \( F(\hat{z})^{-1} \)

DT Frequency Warping

- Simplest mapping \( z = F(\hat{z}) = \frac{\hat{z} - \alpha}{1 - \alpha \hat{z}} \)
  has effect of warping frequency axis:
  \[ \hat{z} = e^{j\hat{\omega}} \Rightarrow z = e^{j\omega} = \frac{e^{j\hat{\omega}} - \alpha}{1 - \alpha e^{j\hat{\omega}}} \]
  \( \alpha > 0 : \) expand HF
  \( \alpha < 0 : \) expand LF

\[ \Rightarrow \tan\left(\frac{\omega}{2}\right) = \frac{1+\alpha}{1-\alpha} \tan\left(\frac{\hat{\omega}}{2}\right) \]
Another Design Example

- Spec:
  - Bandpass, from 800-1600 Hz (SR = 8kHz)
  - Ripple = 1dB, min. stopband atten. = 60 dB
  - 8th order, best transition band
- Use **elliptical** for best performance
- Full design path:
  - design analog LPF prototype
  - analog LPF → BPF
  - CT BPF → DT BPF (Bilinear)

---

Another Design Example

- Or, do it all in one step in Matlab:
  
  ```matlab
  [b,a] = ellip(8,1,60, [800 1600]/(8000/2));
  ```