§ 3.4 Property of Fourier Series

Short-hand Notation :

$$x(t) \xleftrightarrow{\mathcal{F}} X_n$$

$x(t)$: periodic with period T_0 $\qquad$ X_n: Fourier Series

$$\begin{cases} x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t} \\ X_n = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-jn\omega_0 t}\, dt \end{cases}$$

① linearity :

if $x(t) \longleftrightarrow X_n$

$y(t) \longleftrightarrow Y_n$

then $a\,x(t) + b\,y(t) \longleftrightarrow a X_n + b Y_n$

② Time Shifting

if $x(t) \longleftrightarrow X_n$

then $x(t-t_0) \longleftrightarrow e^{-jn(\omega_0 t_0)} X_n$

Note that when a periodic signal is shifted in time, the magnitudes of it $\mathcal{F}$-series coefficients remain the same.

Proof: let $z(t) = x(t-t_0)$

then $Z_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z(t)\, e^{-jn\omega_0 t}\, dt$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t-t_0)\, e^{-jn\omega_0 t}\, dt$$

Let $\tau = t - t_0$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}-t_0}^{\frac{T_0}{2}-t_0} x(\tau)\, e^{-jn\omega_0(\tau+t_0)}\, d\tau$$

$$= e^{-jn\omega_0 t_0}\frac{1}{T_0}\int_{-\frac{T_0}{2}-t_0}^{\frac{T_0}{2}-t_0} x(\tau)\, e^{-jn\omega_0 \tau}\, d\tau = e^{-jn\omega_0 t_0}\underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(\tau)\, e^{-jn\omega_0 \tau}\, d\tau}_{X_n}$$

③ Time Reversal

If $\quad x(t) \longleftrightarrow X_n$

then $\quad x(-t) \longleftrightarrow X_{-n}$

Proof: let $z(t) = x(t)$

then $Z_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z(t)\, e^{-jn\omega_0 t}\, dt$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(-t)\, e^{-jn\omega_0 t}\, dt$$

Let $\tau = -t \quad = \frac{1}{T_0}\left(-\int_{\frac{T_0}{2}}^{-\frac{T_0}{2}} x(\tau)\, e^{jn\omega_0 \tau}\, d\tau\right)$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{jn\omega_0 t}\, dt = X_{-n}$$

$\Longrightarrow$ If $x(t)$ is even $\Longleftrightarrow$ X_n is even

If $x(t)$ is odd $\Longleftrightarrow$ X_n is odd.

④ Time scaling

If $x(t)$ is periodic with period $T_0 \leftrightarrow \omega_0$

then $x(\alpha t)$ is periodic with period T_0/α

$$\Longrightarrow \text{ if } x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$\text{then } x(\alpha t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 \alpha t}$$

$\therefore$ Fourier coefficients remain the same.

But Fourier series representation Has changed because of the change in fundamental freq !

⑤ Multiplication

$$\text{both with period } T_0 \begin{cases} x(t) \longleftrightarrow X_n \\ y(t) \longleftrightarrow Y_n \end{cases}$$

$$\Rightarrow z(t) \triangleq x(t)\,y(t) \longleftrightarrow Z_n = \underbrace{\sum_{l=-\infty}^{\infty} X_l Y_{n-l}}_{\text{discrete convolution}}$$

Proof: $$Z_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z(t)\, e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\,y(t)\, e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{l} X_l e^{jl\omega_0 t} \sum_{k} Y_k e^{jk\omega_0 t}\, e^{-jn\omega_0 t}\,dt$$

$$= \sum_{l}\sum_{k} X_l Y_k \underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j(l+k-n)\omega_0 t}\,dt}_{= \begin{cases} 1 & \text{if } l+k=n \\ 0 & \text{if } l+k\neq n \end{cases}}$$

$$= \sum_{l} X_l Y_{n-l}$$

⑥ Conjugation

If $$x(t) \longleftrightarrow X_n$$
$$x^*(t) \longleftrightarrow X_{-n}^*$$

Proof: let $z(t) = x^*(t)$

$$Z_n = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x^*(t)\, e^{-jn\omega_0 t}\,dt$$

$$= \Big(\underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{jn\omega_0 t}\, dt}_{X_{-n}} \Big)^*$$

$$= X_{-n}^*$$

- If $x(t)$ is real, then $x(t) = x^*(t)$

$$\therefore \quad X_n = X_{-n}^* \quad \text{or} \quad X_{-n} = X_n^*$$

$\Rightarrow |X_{-n}| = |X_n|$ — magnitude is even

$\measuredangle X_{-n} = -\measuredangle X_n$ — phase is odd.

⑦ Parseval's Relation (Theorem)

$$\underbrace{\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt}_{\text{power}} = \sum_{k=-\infty}^{\infty} |X_k|^2$$

power of a periodic signal

= sum of powers in all its harmonic components :

$$\frac{1}{T_0} \int_{-T_0/2}^{\frac{T_0}{2}} |X_k e^{jk\omega_0 t}|^2 dt$$

$$= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |X_k|^2 dt = |X_k|^2$$

Proof :

$$\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 \, dt$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left(\sum_n X_n e^{jn\omega_0 t}\right)\left(\sum_m X_m e^{jm\omega_0 t}\right)^* dt$$

$$= \sum_n \sum_m X_n X_m^* \underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j(n-m)\omega_0 t} \, dt}_{= \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}}$$

$$= \sum_n |X_n|^2$$

⑧ Symmetries

$\underline{Ex}$: $x(t)$ is real with period T &
Fourier series coefficients X_k

Recall : $x(t)$ is real $\Longrightarrow \begin{cases} X_{-n} = X_n^* \\ X_0 \to \text{real} \end{cases}$

① If $x(t)$ is even $\Longrightarrow$ X_n is real & even

② If $x(t)$ is odd $\Longrightarrow \begin{cases} X_n \text{ is imag \& odd} \\ X_0 = 0. \end{cases}$

③ $\underbrace{x_e(t)}_{\text{Even part of } x(t)} \longleftrightarrow \mathrm{Re}\{X_n\}$

④ $x_o(t) \longleftrightarrow j\,\mathrm{Im}\{X_n\}$

Proof : ① $x(t)$ is even

$$\Longleftrightarrow x(t) = x(-t)$$

$$\Updownarrow$$

$$X_n = X_{-n} = X_n^*$$

$X_n = X_n^* \Rightarrow X_n$ is real

$X_n = X_{-n} \Rightarrow X_n$ is even

③ $x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$

$$\longleftrightarrow \frac{1}{2}\left[X_n + X_{-n}\right] = \frac{1}{2}\left[X_n + X_n^*\right]$$

$$= \mathrm{Re}\left[X_n\right]$$