Optimal Flow Control of a Class of Queueing Networks in Equilibrium

AUREL A. LAZAR, MEMBER, IEEE

Abstract —The problem of optimum flow control of a class of queueing systems which appears as a model of datagram and virtual circuit computer communication networks is investigated. This class "intuitively" has the property that by increasing the load on the network, both the average throughput and the average time delay also increase. It is shown that the control that achieves the maximum throughput under a bounded average time delay criterion can be specified by a "window" flow control mechanism (bang-bang control). The window size L, the maximum number of unacknowledged packets in the system, can be easily derived from the preassigned upper bound on the time delay T, the Norton equivalent of the queueing system μ, and the maximum admissible total load on the network c.

I. Introduction

One of the central issues in the design of communication protocols for computer networks is the specification of flow control algorithms. The prevention of throughput degradation due to overload, deadlock avoidance, and fair allocation of network resources are among the main functions of such algorithms [3]. In this paper only the problem of throughput degradation is addressed. In order to do so a suitable model and a relevant optimization criterion for protocol design is considered.

To study the throughput time delay tradeoff, computer communication networks are modeled as queueing networks [12]. Since a key design specification of the protocols is the existence of acknowledgments for packets that have reached their destination, closed queueing networks serve as a model for the study of optimal flow control [6]. Most protocols are designed for unreliable channels in which packets can be lost. To recover from such losses the source retransmits the packets in the event that an acknowledgment has not been received within a certain predetermined time interval [14]. The optimization criterion considered in this paper incorporates this design constraint. First introduced in [7], the criterion adopted maximizes the average throughput under an average time delay and admissible load constraint. Our results show that the window flow control mechanism commonly implemented in practice is optimal with respect to the criterion considered. Thus, our investigations give a justification for the methods already used in practice. In what follows, the "intuitive" arguments presented to motivate our work are formalized.

The problem of optimal flow control of simple queueing systems in equilibrium has recently been investigated [7], [8]. It has been shown that by employing a maximum throughput under a bounded average time delay criterion an $\cdot/M/m$ queue can be optimally controlled with a state-dependent Poissonian flow. In this paper the results previously obtained in [7] and [8] are extended to a class of quasi-reversible queueing networks [4], [15]. This class "intuitively" has the property that by increasing the load on the system, both the average throughput and the average time delay also increase (see Section II).

The class of networks considered arises naturally in problems of end-to-end flow control in computer communication networks [3], [13]. In the queueing model considered in the sequel, both datagram and virtual circuit networks [6] can be accommodated. Interfering traffic from other source–destination pairs is, however, not included. For a fixed source–destination pair the end-to-end behavior of such networks is the same as the one of an equivalent queueing system that can be easily determined [1]. Throughout this paper the Norton equivalent [1] of such networks is assumed to be a queueing system with a state-dependent service rate that is subject to some regularity conditions (see Section II). It is shown that the optimal control of the class of networks investigated is a so-called "window" flow control mechanism [14]. The dependence of the window size on the preassigned maximum time delay, the maximum admissible load c, and the Norton equivalent μ, are explicitly stated.

This paper is structured as follows. In Section II the class of queueing systems under consideration is introduced and the optimization criterion is presented. As in [7], [8] the criterion of maximizing the throughput under a bounded time delay criterion is used. The optimal control that satisfies this criterion is given in Theorem 1 of Section III. This theorem is preceded by a series of lemmas, some of which are of interest in themselves. Finally, the dependence of the maximum throughput on the time delay (the throughput time delay function) is presented for the simple case of a tandem queueing network.

II. The Optimization Criterion

A queueing system as seen by a source–destination pair in a computer communication network together with its acknowledgment path (feedback channel) is schematically depicted in Fig. 1. The upper quasi-reversible queueing network (system) has to be controlled corresponding to a suitable optimality criterion. The closed queueing system in Fig. 1 is assumed to have N packets. k of the total of N packets are assumed to be waiting for service in the upper queueing network. The remaining $N-k$ packets are contained in the lower or "feedback queue."

The class of queueing networks considered throughout this paper has a Norton equivalent [1] (see also, [2] and [11]) with a state-dependent service rate $\mu=(\mu_k)$, $1 \leqslant k \leqslant N$, that satisfies the conditions

$$\mu_i \leqslant \mu_j \leqslant \mu_k \tag{1}$$

and

$$\frac{\mu_j-\mu_i}{j-i} \geqslant \frac{\mu_k-\mu_i}{k-i} \tag{2}$$

for all i, j, and k such that $0 \leqslant i < j < k \leqslant N$.

The set of inequalities above has a very simple analytical (and

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The author is with the Department of Electrical Engineering, Columbia University, New York, NY 10027.

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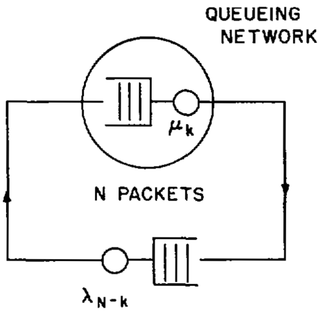

Fig. 1. A queueing model for end-to-end flow control.

geometrical) interpretation: the mapping $k \to \mu_k$, $1 \leqslant k \leqslant N$, is concave nondecreasing. Note that N, the maximum number of packets admitted in the network, is an arbitrarily large integer and $\mu_0 = 0$ in (2) by convention. Obviously, the average service rate of an $\cdot/M/m$ queueing system satisfies the set of inequalities (1) and (2). As another example, consider a link of M tandem queues with constant exponential server δ. The Norton equivalent of the tandem system has a state-dependent server given by [13]

$$\mu_k = \frac{k}{M-1+k}\delta$$

for all k, $1 \leqslant k \leqslant N$. It is easy to see that (μ_k), $1 \leqslant k \leqslant N$, satisfies the inequalities (1) and (2).

Some comments regarding the class of networks defined by (1) and (2) are in order. First, $\mu_k \leqslant \mu_{k+1}$, for all k, $1 \leqslant k \leqslant N$, implies that the throughput is increasing with the number of packets in the system. Second, since inequality (2) implies that $k/\mu_k \leqslant (k+1)/\mu_{k+1}$, for all k, $1 \leqslant k \leqslant N-1$, the time delay of the queueing networks under consideration increases with the number of packets that they contain (see Lemma 2 for more details). Note that the latter inequalities represent the time delay of the upper queueing system in Fig. 1 with "instantaneous" feedback and k and $k+1$ packets, respectively. Note also that the set of inequalities given in (2) can be written as follows:

$$(k-j)\mu_i + (i-k)\mu_j + (j-i)\mu_k \leqslant 0$$

for all i, j, and k such that $0 \leqslant i < j < k \leqslant N$.

The feedback queue in Fig. 1 is assumed to have an exponential server (λ_k), $1 \leqslant k \leqslant N$ that can be controlled. Since there are a maximum of N packets in the above system, the upper network can be seen as a queueing system with a finite buffer size. Without any loss of generality, therefore, the feedback queue models the input stream to a queueing system with a finite buffer size.

The average throughput and the average time delay are given by [5], [13]

$$E\gamma_N = \sum_{k=1}^{N} \mu_k p_k$$

and

$$E\tau_N = \frac{\sum_{k=1}^{N} k p_k}{\sum_{k=1}^{N} \mu_k p_k},$$

respectively, where

$$p_k = \prod_{l=0}^{k-1} \frac{\lambda_{N-l}}{\mu_{l+1}} p_0$$

and

$$p_0 = \frac{1}{1 + \sum_{k=1}^{N} \prod_{l=0}^{k-1} \frac{\lambda_{N-l}}{\mu_{l+1}}}$$

denote the probabilities that the upper queue contains k packets $(1 \leqslant k \leqslant N)$.

Following [7]–[9] a series of definitions will be given first.

Definition 1: $\lambda = (\lambda_k)$, $1 \leqslant k \leqslant N$, will hereafter denote the control.

Definition 2: The class of controls $\lambda = (\lambda_k)$, $1 \leqslant k \leqslant N$, satisfying the peak constraint

$$0 \leqslant \lambda_k \leqslant c$$

for all k, $1 \leqslant k \leqslant N$, where c, $c \in R_+$, is a constant is called admissible.

Definition 3: The control $\lambda = (\lambda_k)$, $1 \leqslant k \leqslant N$, is said to be optimum over the class of admissible controls for a given T, $T \in R_+$, if the maximum

$$\max_{E\tau_N \leqslant T} E\gamma_N$$

is achieved.

Definition 4: The mapping F: $R_+ \to R_+$ given by

$$F(T) = \max_{E\tau_N \leqslant T} E\gamma_N$$

is called the throughput time delay function.

Let x_k denote the expression

$$x_k = \prod_{l=0}^{k-1} \frac{\lambda_{N-l}}{\mu_{l+1}} = \frac{p_k}{p_0}$$

for all k, $1 \leqslant k \leqslant N$. Thus,

$$\sum_{k=1}^{N} x_k = \sum_{k=1}^{N} \frac{p_k}{p_0} = \frac{1-p_0}{p_0} = \frac{1}{p_0} - 1$$

and

$$p_0 = \frac{1}{1 + \sum_{k=1}^{N} x_k}.$$

Lemma 1: $\lambda = (\lambda_k)$, $1 \leqslant k \leqslant N$, is optimum in the class of admissible controls for a given T, $T \in R_+$, if it achieves the maximum

$$F(T) = \max_{\sum_{k=1}^{N}(k-\mu_k T)x_k \leqslant 0} \frac{\sum_{k=1}^{N} \mu_k x_k}{1 + \sum_{k=1}^{N} x_k} \tag{3}$$

where

$$0 \leqslant x_k \leqslant \prod_{l=0}^{k-1} \frac{c}{\mu_{l+1}} = \rho_k$$

for all k, $1 \leqslant k \leqslant N$.

Proof: Since the condition $E\tau_N \leqslant T$ is equivalent to

$$\sum_{k=1}^{N} (k - \mu_k T) x_k \leqslant 0$$

and

$$E\gamma_N = p_0 \cdot \sum_{k=1}^{N} \mu_k x_k = \frac{\sum_{k=1}^{N} \mu_k x_k}{1 + \sum_{x=1}^{N} x_k}$$

the optimum control $\lambda = (\lambda_k)$, $1 \leqslant k \leqslant N$, for a given T, $T \in R_+$, achieves

$$\max_{\sum_{k=1}^{N}(k-\mu_k T)x_k \leqslant 0} \frac{\sum_{k=1}^{N} \mu_k x_k}{1 + \sum_{k=1}^{N} x_k}. \qquad \square$$

III. The Optimal Control

Our main result in this section is given by Theorem 1. Several results will first be proved in order to simplify its proof. In Lemmas 2 and 3 the maximum time delay with L packets in the system (see Fig. 1) is derived and shown to increase with the number of packets. In Lemmas 4 and 5 some purely technical results needed for the proof of Theorem 1 are presented.

Lemma 2: The maximum time delay that can be achieved with L packets in the system is given by

$$T_{\max}^{(L)} = \frac{\sum_{k=1}^{L} k\rho_k}{\sum_{k=1}^{L} \mu_k \rho_k} \leqslant \frac{L}{\mu_L}.$$

Proof: Let us first show that

$$\frac{\sum_{k=1}^{L} k x_k}{\sum_{k=1}^{L} \mu_k x_k} \leqslant \frac{\sum_{k=1}^{L} k\rho_k}{\sum_{k=1}^{L} \mu_k \rho_k}$$

or

$$\sum_{k=1}^{L} \sum_{l=1}^{L} x_k \rho_l (k\mu_l - \mu_k l) \leqslant 0.$$

Due to the symmetry in k and l and the fact that

$$\frac{k}{\mu_k} \leqslant \frac{l}{\mu_l} \quad \text{and} \quad \frac{x_l}{x_k} \leqslant \frac{\rho_l}{\rho_k},$$

for all $k \leqslant l$, the expression on the left-hand side in the above inequality is negative.

Note that

$$\frac{\sum_{k=1}^{L} k\rho_k}{\sum_{k=1}^{L} \mu_k \rho_k} \leqslant \frac{L}{\mu_L}$$

is equivalent to

$$\sum_{k=1}^{L} \rho_k (k\mu_L - L\mu_k) \leqslant 0$$

which is true since

$$\frac{k}{\mu_k} \leqslant \frac{L}{\mu_L},$$

for all $k \leqslant L$. Equality can be only achieved in the $M/M/L$ case, i.e., iff k/μ_k is a constant for all k, $1 \leqslant k \leqslant L$. $\square$

Lemma 3: The maximum time delay of a passive queueing system described by (1) and (2) increases with the number of packets, i.e.,

$$T_{\max}^{(L)} \leqslant T_{\max}^{(L+1)},$$

for all L, $L < N$.

Proof: We have to show that

$$\frac{\sum_{k=1}^{L} k\rho_k}{\sum_{k=1}^{L} \mu_k \rho_k} \leqslant \frac{\sum_{k=1}^{L+1} k\rho_k}{\sum_{k=1}^{L+1} \mu_k \rho_k}.$$

After some algebraic manipulations this inequality can be reduced to

$$\sum_{k=1}^{L} \rho_k [k\mu_{L+1} - (L+1)\mu_k] \leqslant 0$$

which is true since

$$\frac{k}{\mu_k} \leqslant \frac{L+1}{\mu_{L-1}},$$

for all $k \leqslant L$. $\square$

Remark: If there is an integer L such that

$$T_{\max}^{(L)} = T_{\max}^{(L+1)}$$

then k/μ_k is a constant for all k, $1 \leqslant k \leqslant L+1$. Such a situation occurs, for example, in an $M/M/L+1$ queueing system [8].

The method used to prove our main result in Theorem 1 is based on a majorization argument. To find the throughput time delay function [see (3)] an achievable upper bound will first be derived. In order to do so, the achievable upper bounds for the expressions $\sum_{k=1}^{N} \mu_k x_k$ and $\sum_{k=1}^{N} x_k$ that appear in (3) are obtained in Lemma 4. In Lemma 5 an achievable upper bound for the throughput of Norton's equivalent is given. The proofs of Lemmas 4 and 5 can be skipped, if desired, without any loss of continuity.

Lemma 4: Let us assume that $E\tau_N \leqslant T$ and $T < L/\mu_L$ with $2 \leqslant L \leqslant N$. Then:

$$\sum_{k=1}^{N} \mu_k x_k \leqslant -\frac{1}{L - \mu_L T} \sum_{k=1}^{L-1} (k\mu_L - L\mu_k) x_k \tag{4}$$

and

$$\sum_{k=1}^{N} x_k \leqslant \frac{1}{L - \mu_L T} \sum_{k=1}^{L-1} [L - k - (\mu_L - \mu_k)T] x_k \tag{5}$$

and equality can be achieved if

$$x_k = 0,$$

for all k, $L+1 \leqslant k \leqslant N$, and

$$x_L = \frac{1}{L - \mu_L T} \sum_{k=1}^{L-1} (\mu_k T - k) x_k.$$

Proof: The linear constraint on the time delay $E\tau_N \leqslant T$ implies that

$$x_L \leqslant -\frac{1}{L-\mu_L T}\sum_{\substack{k=1\\k\neq L}}^{N}(k-\mu_k T)x_k.$$

Therefore, by simple addition of the same expression on both sides of the above inequality, we obtain

$$\sum_{k=1}^{N}\mu_k x_k \leqslant -\frac{1}{L-\mu_L T}\sum_{\substack{k=1\\k\neq L}}^{N}(k\mu_L - L\mu_k)x_k$$

and

$$\sum_{k=1}^{N}x_k \leqslant \frac{1}{L-\mu_L T}\sum_{\substack{k=1\\k\neq L}}^{N}[L-k-(\mu_L-\mu_k)T]x_k.$$

Inequalities (4) and (5) can now easily be obtained since

$$\frac{k}{\mu_k} \geqslant \frac{L}{\mu_L}$$

and

$$T < \frac{L}{\mu_L} \leqslant \frac{k-L}{\mu_k-\mu_L}$$

for all k, $L+1 \leqslant k \leqslant N$ [a little thought shows that the latter inequality can be obtained without using (1)]. □

Lemma 5: Let $E\tau_N \leqslant T$ and $T < L/\mu L$ with $2 \leqslant L \leqslant N$. Then,

$$\left\{\sum_{k=1}^{N}\mu_k x_k\right\}\left\{1+\sum_{k=1}^{N}x_k\right\}^{-1} \leqslant \left\{\sum_{k=1}^{L-1}(L\mu_k - k\mu_L)x_k\right\} \cdot \left\{L-\mu_L T + \sum_{k=1}^{L-1}[L-k-(\mu_L-\mu_k)T]x_k\right\}^{-1} \tag{6}$$

and equality can be achieved if $x_k = 0$, for all k, $L+1 \leqslant k \leqslant N$, and

$$x_L = \frac{1}{L-\mu_L T}\sum_{k=1}^{L-1}(\mu_k T - k)x_k.$$

Proof: To prove the assertion above, it is enough to show that

$$\left\{\sum_{k=1}^{N}\mu_k x_k\right\}\left\{1+\sum_{k=1}^{N}x_k\right\}^{-1} \leqslant \left\{\sum_{k=1}^{N}(L\mu_k - k\mu_L)x_k\right\} \cdot \left\{L-\mu_L T + \sum_{k=1}^{L-1}[L-k-(\mu_L-\mu_k)T]x_k\right\}^{-1}$$

or

$$(L-\mu_L T)\left\{\sum_{k=1}^{N}\mu_k x_k - \frac{1}{L-\mu_L T}\sum_{k=1}^{N}(L\mu_k - k\mu_L)x_k\right\} + \sum_{k=1}^{N}\sum_{l=1}^{L-1}\mu_k[L-l-(\mu_L-\mu_l)T]x_k x_l - \sum_{k=1}^{N}\sum_{l=1}^{L-1}(L\mu_l - l\mu_L)x_k x_l \leqslant 0$$

and equality can be achieved if $x_k = 0$, for all k, $L+1 \leqslant k \leqslant N$, and

$$x_L = \frac{1}{L-\mu_L T}\sum_{k=1}^{L-1}(\mu_k T - k)x_k.$$

The latter inequality can be rearranged as

$$(L-\mu_L T)\left\{\sum_{k=1}^{N}\mu_k x_k - \frac{1}{L-\mu_L T}\sum_{k=1}^{N}(L\mu_k - k\mu_L)x_k\right\} + \sum_{k=L}^{N}\sum_{l=1}^{L-1}[(k-L)\mu_l + (l-k)\mu_L + (L-l)\mu_k)]x_k x_l + \sum_{l=1}^{L-1}(\mu_L-\mu_l)x_l\left(\sum_{k=1}^{N}kx_k - \sum_{k=1}^{N}\mu_k x_k\cdot T\right) \leqslant 0. \tag{7}$$

Due to Lemma 4, the first expression in the above inequality is negative. Similarly, the second and the third expressions on the left-hand side of inequality (7) are also negative since

$$(k-L)\mu_l + (l-k)\mu_L + (L-l)\mu_k \leqslant 0$$

for all l, k, $l < L \leqslant k$, and

$$\left\{\sum_{k=1}^{N}kx_k\right\}\left\{\sum_{k=1}^{N}\mu_k x_k\right\}^{-1} \leqslant T$$

by assumption. It remains as a simple exercise to verify that equality in (7) can be achieved if $x_k = 0$, for all k, $L+1 \leqslant k \leqslant N$, and

$$x_L = \frac{1}{L-\mu_L T}\sum_{k=1}^{L-1}(\mu_k T - k)x_k. \qquad \square$$

We are now in the position to prove our main result concerning the optimal control of queueing networks having a Norton equivalent μ that belongs to the class defined by (1) and (2). Since the time delay is increasing with the number of packets in the network (see Lemma 2) there exists an integer L, $L \leqslant N$, such that for a given maximum time delay T, $T_{\max}^{(L-1)} < T \leqslant T_{\max}^{(L)}$. Since $T > T_{\max}^{(L-1)}$ we expect that the control of the first $L-1$ packets will be $\lambda_k = c$ for all k, $1 \leqslant k \leqslant L-1$. The remaining "gap" in the average time delay $T - T_{\max}^{(L-1)}$ will be "filled" by the delay caused by the Lth packet. The control of the Lth packet has yet to be determined. In the following we will assume that $T_{\max}^{(L)} < L/\mu_L$. The case $T_{\max}^{(L)} = L/\mu_L$ has already been treated in [8] (see also the Remark following Theorem 1).

Theorem 1: Given that $T_{\max}^{(L-1)} < T \leqslant T_{\max}^{(L)}$, $2 \leqslant L \leqslant N$, the optimal control of a passive queueing network with a maximum of N packets in the system is given by

$$\lambda_k = \begin{cases} c & N-L+2 \leqslant k \leqslant N \\ \lambda_{N-L+1} & k = N-L+1 \\ 0 & 1 \leqslant k \leqslant N-L \end{cases}$$

where

$$\lambda_{N-L+1} = \frac{1}{L/\mu_L - T}\sum_{l=1}^{L-1}(\mu_l T - l)\rho_l/\rho_{L-1}. \tag{8}$$

Finally,

$$F(T) = \left\{\sum_{k=1}^{L-1}(L\mu_k - k\mu_L)\rho_k\right\} \cdot \left\{L-\mu_L T + \sum_{k=1}^{L-1}[L-k-(\mu_L-\mu_k)T]\rho_k\right\}^{-1}. \tag{9}$$

Proof: To prove (9) in view of Lemmas 1 and 5, let us first show that

$$\left\{\sum_{k=1}^{L-1}(L\mu_k-k\mu_L)x_k\right\}\cdot\left\{L-\mu_L T+\sum_{k=1}^{L-1}[L-k-(\mu_L-\mu_k)T]x_k\right\}^{-1}\leqslant\left\{\sum_{k=1}^{L-1}(L\mu_k-k\mu_L)\rho_k\right\}\cdot\left\{L-\mu_L T+\sum_{k=1}^{L-1}[L-k-(\mu_L-\mu_k)T]\rho_k\right\}^{-1}$$

or

$$(L-\mu_L T)\cdot\sum_{k=1}^{L-1}(L\mu_k-k\mu_L)(x_k-\rho_k)+\sum_{k=1}^{L-1}\sum_{l=1}^{L-1}(L\mu_k-k\mu_L)[L-l-(\mu_L-\mu_l)T]x_k\rho_l-\sum_{k=1}^{L-1}\sum_{l=1}^{L-1}[L-k-(\mu_L-\mu_k)T](L\mu_l-l\mu_L)x_k\rho_l\leqslant 0.$$

After some simple algebraic manipulations, this inequality becomes

$$\sum_{k=1}^{L-1}(L\mu_k-k\mu_L)(x_k-\rho_k)+\sum_{k=1}^{L-1}\sum_{l=1}^{L-1}[(k-L)\mu_l+(L-l)\mu_k+(l-k)\mu_L]x_k\rho_l\leqslant 0.$$

Both expressions on the left-hand side are negative since

$$x_k\leqslant\rho_k$$

for all k, $1\leqslant k\leqslant L-1$, and

$$(k-L)\mu_l+(L-l)\mu_k+(l-k)\mu_L\geqslant 0,$$

and finally

$$\frac{x_k}{x_l}\leqslant\frac{\rho_k}{\rho_l}$$

for all l, k, $l\leqslant k\leqslant L-1$.

Hence, the maximum throughput given by (9) is achieved if $x_k=\rho_k$, for all k, $1\leqslant k\leqslant L-1$, $x_k=0$, for all k, $L+1\leqslant k\leqslant N$ and

$$x_L=\frac{1}{L-\mu_L T}\sum_{l=1}^{L-1}(\mu_l T-l)\rho_l.$$

The values obtained above for (x_k), $1\leqslant k\leqslant N$, also satisfy the time delay constraint $E\tau_N\leqslant T$ (see Lemma 5). Therefore, it remains only to verify that

$$0<x_L\leqslant\rho_L$$

which can be written as

$$0<\sum_{l=1}^{L-1}(\mu_l T-l)\rho_l\leqslant(L-\mu_L T)\rho_L$$

or

$$\left\{\sum_{l=1}^{L-1}l\rho_l\right\}\left\{\sum_{l=1}^{L-1}\mu_l\rho_l\right\}^{-1}<T\leqslant\left\{\sum_{l=1}^{L}l\rho_l\right\}\left\{\sum_{l=1}^{L}\mu_l\rho_l\right\}^{-1},$$

that is equivalent to

$$T_{\max}^{(L-1)}<T\leqslant T_{\max}^{(L)}. \tag{10}$$

Remark: As in the $M/M/m$ case [7], since the lower queue will contain at all times at least $N-L$ packets, the optimum control can also be achieved by a control scheme using a total of only $N=L$ packets. We require that

$$\lambda_k=\begin{cases}c & 2\leqslant k\leqslant L\\ \lambda_1 & k=1\end{cases} \tag{11}$$

where λ_1 is exactly the right-hand side of (8). Therefore, the optimal control is a window type mechanism. The window size L can be easily derived from the maximum time delay of the system by using the inequalities (10). Naturally, the number of packets will depend on the maximum offered load (or line capacity into the queueing network).

In Theorem 1 we have implicitly assumed the existence of an integer L such that [see the inequalities (10)]

$$T_{\max}^{(L-1)}<T_{\max}^{(L)}.$$

If, on the other hand, there is an integer $L_{\max}$ such that

$$T_{\max}^{(L-1)}=T_{\max}^{(L)}$$

and

$$T_{\max}^{(L-1)}<T_{\max}^{(L)}$$

for all L, $L>L_{\max}$, then the optimum control remains the same as the one presented above as long as $L>L_{\max}$. For $L\leqslant L_{\max}$ the optimum control will be similar to the one obtained in [8] for the $M/M/\infty$ queue. Note that the results of [8] for the optimal flow control of an $M/M/m$ queueing system can be directly derived from Theorem 1 by setting $\mu_k=\min(k,m)$ for all k, $1\leqslant k\leqslant N$.

Except for at most one packet, the results obtained [see also (11)] indicate that all packets should be "injected" in the network at the maximum rate c. The different rate of the Lth packet would increase the complexity in implementation. This difficulty can be easily avoided in practice by only considering the discrete number of admissible delays $T=T_{\max}^{(L)}$, for all L, $1\leqslant L\leqslant N$. For such admissible delays, all L packets will be served at the same rate c. The above analysis motivates the introduction of the following.

Definition 5: The tuple $(T_{\max}^{(L)}, F(T_{\max}^{(L)}))$ is called an operating point. The set of tuples $\{(T_{\max}^{(L)}, F(T_{\max}^{(L)}))\}$, $L\leqslant N$, is called the set of operating points.

The set of operating points is completely specified by the following.

Corollary: If $T=T_{\max}^{(L)}$, then

$$\left(T_{\max}^{(L)}, F\left(T_{\max}^{(L)}\right)\right)=\left(\frac{\sum_{k=1}^{L}k\rho_k}{\sum_{k=1}^{L}\mu_k\rho_k}, \frac{\sum_{k=1}^{L}\mu_k\rho_k}{1+\sum_{k=1}^{L}\rho_k}\right)$$

for all L, $L\leqslant N$.

Proof: Since $T=T_{\max}^{(L)}$ we have

$$x_L=\frac{1}{L-\mu_L T}\sum_{l=1}^{L-1}(\mu_l T-l)\rho_l=\rho_L$$

and therefore

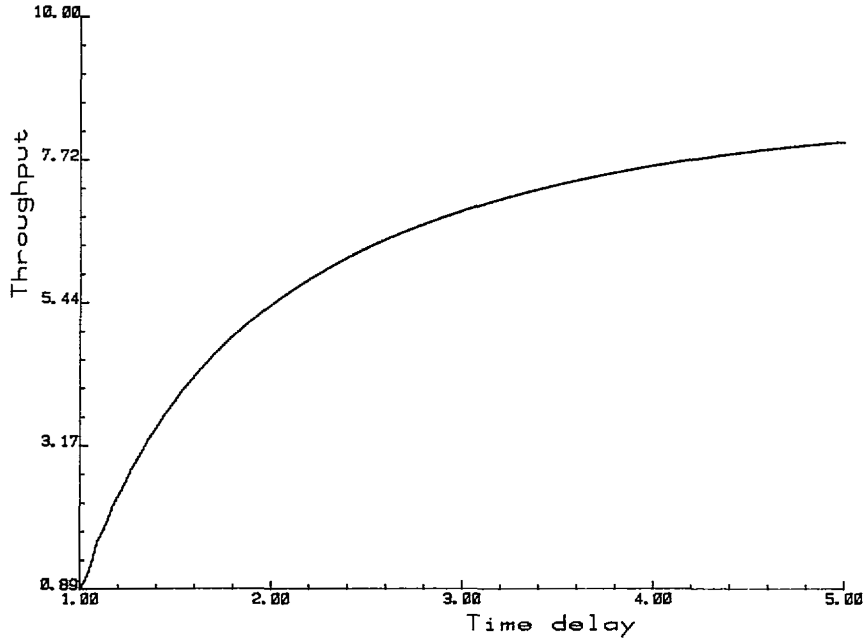

Fig. 2. The throughput time delay function ($M=10$, $\mu=10$, $c=8$).

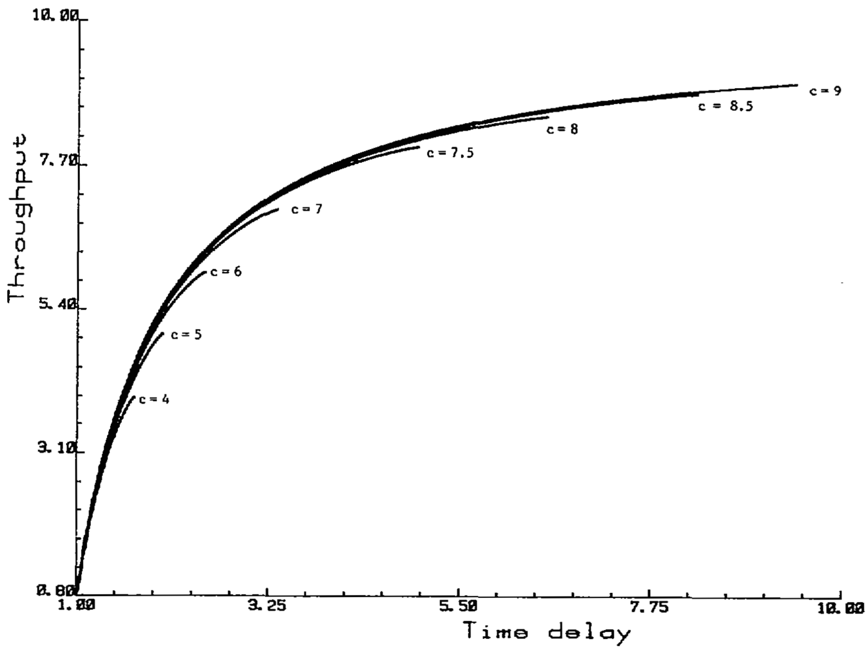

Fig. 3. The dependence of the throughput time delay function on the parameter c.

$$F\left(T_{\max}^{(L)}\right)=\frac{\sum_{k=1}^{L}\mu_k\rho_k}{1+\sum_{k=1}^{L}\rho_k}. \qquad \square$$

Lemma 6: The throughput time delay function is continuous nondecreasing on R_+. In addition, for all T, $T_{\max}^{(L-1)}<T\leqslant T_{\max}^{(L)}$, and $L\leqslant N$, F is convex.

Proof: The proof is left to the reader as a simple exercise.

The throughput time delay function of a tandem queueing system containing $M=10$ queues, each with an exponential service time distribution having parameter $\mu=10$ and a maximum capacity input line $c=8$, is graphically shown in Fig. 2. Fig. 3 depicts the throughput time delay function of the same tandem network for various values of the parameter c. The "overall concave" behavior of the throughput time delay function can also be easily proven.

IV. Conclusion

The main contribution of this work lies in identifying a class of queueing systems that represent an analytically tractable model for optimal end-to-end flow control in computer communication networks. For this class of queueing networks the window flow control maximizes the throughput under an average time delay and admissible load constraint. The window size can easily be derived from the upper bound on the average time delay T, the maximum admissible load c, and the Norton equivalent of the network μ. The results obtained suggest that the window flow control mechanism currently employed in computer communication networks has some very desirable properties. Recently, it has been shown that the Jacksonian network is a member of the class of networks studied in this paper [10].

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Aurel A. Lazar (S'77–M'80) was born in Zalau/Transylvania, Romania, on January 30, 1950. He received the Diplom-Ingenieur degree in communications (nachrichtentechnik) from the Technische Hochschule Darmstadt, Darmstadt, Germany, in 1976 and the Ph.D. degree in information sciences and systems from Princeton University, Princeton, NJ, in 1980.

During his studies he was involved in research on ADPCM coding, adaptive equalization, and information processing with point process observations. He is now an Assistant Professor of Electrical Engineering at Columbia University, New York, NY. His research interests include computer communication and distributed sensor networks, estimation and detection theory, and auditory neural coding.