# The Progressive Second Price Auction Mechanism for Network Resource Sharing*

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## 1 Introduction

In the emerging multiservice communication networks (ATM, Next-Generation Internet), traditional approaches to pricing are not viable. In telephony, the resources allocated to a call are fixed, and usage prices are based on the predictability of the number of active calls at any given moment. But the wide and rapidly evolving range of applications (including some which adapt to resource availability) in the new networks makes demand much more difficult to predict. On the other hand , the current Internet practice of pricing by the physical capacity decouples actual use of resources from the monetary charges, making the network vulnerable to the well-known "tragedy of the commons". Thus there is a need to develop new approaches to pricing of network resources, which has led to much research in recent years (see [11, 8, 7, 3, 12, 9, 2] for a representative sample).

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In [10], we present the Progressive Second Price (PSP) auction, an efficient mechanism for allocation of variable-size shares of a resource among multiple users. The PSP rule generalizes Vickrey ("second-price") auctions [13] which are for non-divisible objects, and bears some similarity to Clarke-Groves mechanisms [1, 5]. The difference with the latter being that for practical reasons, we reduce the message (bid) space to two dimensions (price and quantity), rather than the infinite dimensional space of valuation functions (or demand schedules) which is required in the direct-revelations mechanisms. Player $i$'s bid is $s_i = (q_i, p_i) \in \mathcal{S}_i = [0, Q] \times [0, \infty)$, meaning he would like a quantity $q_i$ at a *unit* price $p_i$. A bid profile is $s = (s_1, \ldots, s_I)$. The auctioneer follows an auction rule $A$ to respond with an allocation $A(s) = (a(s), c(s))$, where $a_i(s)$ and $c_i(s)$ are respectively the quantity allocated to and the *total* charge paid by player $i$.

The PSP allocation rule is:

$$
\begin{aligned}
a_i(s) &= q_i \wedge \underline{Q}_i(p_i, s_{-i}) && (1) \\
c_i(s) &= \sum_{j \neq i} p_j \left[ a_j(0; s_{-i}) - a_j(s_i; s_{-i}) \right] && (2)
\end{aligned}
$$

where $\wedge$ means taking the minimum,

$$
\underline{Q}_i(y, s_{-i}) = \left[ Q - \sum_{p_k \geq y, k \neq i} q_k \right]^+, \qquad (3)
$$

and $Q$ is the total available quantity of resource. $\underline{Q}_i(y, s_{-i})$ is the quantity available for player $i$ at a bid price of $y$. The rule is computationally simple – $O(I^2)$ where $I$ is the number of players – and can thus be used in real-time dynamic auctioning. In [10], we show that under elastic demand (concave valuations), analyzed as a complete information game, the PSP auction is incentive compatible and stable, in that it has a "truthful" $\epsilon$-Nash equilibrium where all players bid at prices equal to their marginal valuation of the resource, for any seller reserve price $p_0 > 0$. PSP is efficient in that the equilibrium allocation maximizes total user value to within $O(\sqrt{\epsilon})$. The parameter $\epsilon$ has a natural interpretation as a bid fee, and allows a manager to directly trade-off engineering and economic efficiency (measured respectively by convergence time and total user value).

In this paper, we show that the equilibrium holds when PSP is applied by independent resource sellers on each link of a network with arbitrary topology, with users having arbitrary but fixed routes. In this network case, the distributed mechanism has a further incentive compatibility in that submitting the same bid at all links along the route is an optimal strategy for each user, regardless of other players' actions. Thus, PSP consitutes a stable decentralized mechanism for allocating and pricing capacity for virtual paths (VPs) and virtual private networks (VPN), and is applicable to programmable ATM networks [4].

# 2 Decentralized PSP Auctioning of Networked Resources

## 2.1 Formulation

In this section, we extend the formulation of [10] to the network case, where there is a set of resources $\mathcal{L} = \{1, \ldots, L\}$, of which the quantities are $Q^1, \ldots, Q^L$, and as before, a set of players $\mathcal{I} = \{1, \ldots, I\}$.

A basic goal is that the mechanism be decentralized in that the allocations at any link depend only on local information: the resources available at that link and the bids for that link only. This makes the mechanism applicable to cases where the various resources being auctioned may be owned by different entities. Each player is responsible for coordinating (or not) her bids at the different links on her route in such a way that maximizes her utility.

Let $\mathcal{Q}^l = [0, Q^l]$, and $\mathcal{Q} = \prod_{l \in \mathcal{L}} \mathcal{Q}^l$. Player $i$'s bid is now $s_i = (s_i^1, \ldots, s_i^L) \in \mathcal{S}_i = \prod_{l \in \mathcal{L}} \mathcal{S}_i^l$, where $s_i^l = (q_i^l, p_i^l) \in \mathcal{S}_i^l = \mathcal{Q}^l \times [0, \infty)$ is the bid for resource $l \in \mathcal{L}$. At each link $l \in \mathcal{L}$, we have an allocation rule $A^l$, which maps a profile $s^l \in \mathcal{S}^l = \prod_{i \in \mathcal{I}} \mathcal{S}_i^l$ to an allocation $A^l(s^l) = (a^l(s^l), c^l(s^l))$.

Player $i$'s type includes a *route*[1], $r_i \subset \mathcal{L}$. We will assume that players only care about the end-to-end "thickness" of their allocated "pipe" (which is given by the thinnest link allocation) and the total charge. Thus player $i$ has a *valuation* of the resource $\theta_i(.)$. Thus, for a bid profile of $s$, under allocation rule $A$, player $i$ getting an allocation $A_i(s) = (a_i(s), c_i(s))$ has the quasilinear utility

$$u_i(s) = \theta_i(\min_{l \in r_i} a_i^l(s)) - c_i(s), \tag{4}$$

where

$$c_i(s) = \sum_{l \in \mathcal{L}} c_i^l(s).$$

[1]Our analysis will not require that $r_i$ form a continuous path, or any specific type of subgraph – "route" means any arbitray subset of links.

In addition, the player can be constrained by a *budget* $b_i \in [0, \infty]$, so the bid $s_i$ must satsify

$$S_i(s_{-i}) = \{s_i \in \mathcal{S}_i : c_i(s_i; s_{-i}) \leq b_i\}. \tag{5}$$

**Remark**: In reality, routing itself is a competitive game, and the decentralized nature of the auction makes it possible for players to make the route part of their strategy and thus vary it in response to other users' actions. In our analysis however, we assume players have obtained a (fixed) route before entering the auction game. In the broader context, the auction game may be nested within a larger game which includes routing.

## 2.2 Equilibrium of Networked PSP Auctions

Assume that the allocation at each link $l \in \mathcal{L}$ is performed by a PSP rule, i.e. $A^l$ satifies (1) and (2). Assume further that the demand is elastic:

**Assumption 1** *For any $i \in \mathcal{I}$,*

- $\theta_i(0) = 0$,
- $\theta_i$ *is differentiable,*
- $\theta_i' \geq 0$*, non-increasing and continuous*
- $\exists \gamma_i > 0,\ \forall z \geq 0,\ \theta_i'(z) > 0 \Rightarrow \forall \eta < z, \theta_i'(z) \leq \theta_i'(\eta) - \gamma_i(z - \eta).$

The key property in the analysis of the network case is that, given a fixed opponent profile, a player cannot do better than place *consistent* bids, i.e, the same bid at all the links on her path and bid zero on all links not in her path.

For each $i \in \mathcal{I}$, we define

$$\begin{array}{rccl} x_i : & \mathcal{S} & \longrightarrow & \mathcal{S}_i \\ & s & \longmapsto & x_i(s) = (z_i, y_i), \end{array}$$

where for $1 \leq l \leq L$,

$$\begin{array}{rcl} z_i^l & = & 1_{r_i}(l) \min_{m \in r_i} a_i^m(s), \\ y_i^l & = & 1_{r_i}(l) \max_{m \in r_i} p_i^m. \end{array}$$

Define also

$$Q_i^l(y, s_{-i}^l) = \lim_{\eta \searrow y} \underline{Q}_i^l(\eta, s_{-i}^l),$$

and

$$P_i^l(z, s_{-i}) = \inf\left\{y \geq 0 : Q_i^l(y, s_{-i}) \geq z\right\}. \tag{6}$$

where for each $l \in \mathcal{L}$, $\underline{Q}_i^l$ is defined by 3.

**Proposition 1** *$\forall s \in \mathcal{S}$, and $i \in \mathcal{I}$,*

$$u_i(x_i(s); s_{-i}) \geq u_i(s).$$

*Moreover ,*

$$s_i \in S(s_{-i}) \Rightarrow x_i(s) \in S(s_{-i}).$$

**Proof:** First, we prove that, leaving bid prices unchanged, there is no loss of utility for a player who reduces the bid quantities to $z_i$, i.e.

$$u_i((z_i, p_i); s_{-i}) \geq u_i(s).$$

For any $l \in r_i$, $z_i^l \leq a_i^l(s) \leq Q_i^l(p_i^l, s^l)$, therefore $a_i^l((z_i, p_i); s_{-i}) = z_i^l \wedge Q_i^l(p_i^l, s_{-i}^l) = z_i^l = min_{m \in r_i} a_i^m(s)$. Thus,

$$\begin{aligned}
& u_i((z_i, p_i); s_{-i}) - u_i(s) \\
&= \theta_i(min_{m\in r_i} a_i^m((z_i, p_i); s_{-i})) - \theta_i(min_{m\in r_i} a_i^m(s)) - c_i((z_i, p_i); s_{-i}) + c_i(s) \\
&= 0 - c_i((z_i, p_i); s_{-i}) + c_i(s) \\
&= \sum_{l \in \mathcal{L}} \int_{a_i^l((z_i,p_i);s_{-i})}^{a_i^l(s)} P_i^l(z, s_{-i})\, dz \\
&\geq 0,
\end{aligned}$$

since $P_i^l \geq 0$ and $a_i^l((z_i, p_i); s_{-i}) \leq a_i^l(s)$.

Second, for any $l \in r_i$, $y_i^l \geq p_i^l$, hence $Q_i^l(y_i^l, s^l) \geq Q_i^l(p_i^l, s^l) \geq z_i^l$. Thus, $a_i^l((z_i^l, y_i^l); s_{-i}^l) = z_i^l = a_i^l((z_i, p_i); s_{-i})$. Now since $z_i^l = 0$ for $l \notin r_i$, we have

$$\begin{aligned}
c_i((z_i, y_i); s_{-i}) &= \sum_{l \in \mathcal{L}} \int_0^{a_i^l((z_i,y_i);s_{-i})} P_i^l(z, s_{-i})\, dz \\
&= \sum_{l \in r_i} \int_0^{a_i^l((z_i,y_i);s_{-i})} P_i^l(z, s_{-i})\, dz \\
&= \sum_{l \in r_i} \int_0^{a_i^l((z_i,p_i);s_{-i})} P_i^l(z, s_{-i})\, dz \\
&= c_i((z_i, p_i); s_{-i}),
\end{aligned}$$

hence

$$u_i((z_i, y_i); s_{-i}) = u_i((z_i, p_i); s_{-i}),$$

which completes the proof of the first statement.

Now $s_i \in S(s_{-i}) \Rightarrow b_i \geq c_i(s) \geq c_i((z_i, y_i); s_{-i}) = c_i((z_i, p_i); s_{-i}) \Rightarrow (z_i, p_i) \in S(s_{-i})$. $\square$

Thanks to Proposition 1, we can restrict our attention to consistent strategies only, and still have feasible best replies[2]. This forms a "consistent" embedded game with feasible sets replaced by the consistent strategy set obtained by applying $x_i$ to the feasible strategy set

$$\tilde{S}_i(s_{-i}) = x_i(S(s_{-i}) \times \{s_{-i}\}).$$

Define for $\forall y, z \geq 0$, $\forall s \in \mathcal{S}$, $\forall i \in \mathcal{I}$,

$$P_i^l(z, s_{-i}^l) = \inf \left\{ \eta \geq 0 : Q_i^l(\eta, s_{-i}) \geq z \right\}. \tag{7}$$

Let

$$\begin{aligned}
\tilde{P}_i(z, s_{-i}) &= \sum_{l \in r_i} P_i^l(z, s_{-i}^l), \\
\tilde{Q}_i(y, s_{-i}) &= \sup\{z \in \bigcap_{l \in r_i} \mathcal{Q}^l : \tilde{P}_i(z, s_{-i}) < y\}, \\
\underline{\tilde{Q}}_i(y, s_{-i}) &= \min_{l \in r_i} \underline{Q}^l_i(y, s_{-i}^l), \\
\tilde{a}_i(s) &= q_i^1 \wedge \underline{\tilde{Q}}_i(p_i^1, s_{-i}), \\
\tilde{c}_i(s) &= \int_0^{\tilde{a}_i(s)} \tilde{P}_i(z, s_{-i})\, dz.
\end{aligned}$$

**Lemma 1** $\forall s_{-i} \in \mathcal{S}_{-i}, \forall s_i \in \tilde{S}(s_{-i}), \forall l \in r_i$,

$$\tilde{Q}_i(y, s_{-i}) = \lim_{\eta \searrow y} \underline{\tilde{Q}}_i(\eta, s_{-i}),$$

$$\tilde{a}_i(s) = \min_{l \in r_i} a_i^l(s),$$

$$\tilde{c}_i(s) = c_i(s),$$

*and*

$$u_i(s) = \theta_i(\tilde{a}_i(s)) - \tilde{c}_i(s).$$

[2] If $\theta_i' > 0$, an even stronger statement holds: a bid can be a best reply only if it results in the same quantity allocation at all the links in the route.

**Proof:** Follows trivially from the definitions and the fact that $s_i \in \tilde{S}_i(s_{-i}) \Rightarrow q_i^l = a_i^l(s) = q_i^1, \forall l \in r_i$. $\square$

Now within the feasible sets, the embedded game is identical to the single node game, with all elements being replaced with the $\tilde{}$ version. Thus the following result from [10] holds: for a given opponent profile $s_{-i}$, an $\epsilon$-best reply for player $i$ is to bid at a price equal to the marginal valuation, i.e. set $p_i = \theta_i'(q_i)$. Formally, let $\mathcal{T}_i = \{s_i \in \mathcal{S}_i : p_i = \theta_i'(q_i)\}$, the (unconstrained) set of player $i$'s truthful bids, and $\mathcal{T} = \prod_i \mathcal{T}_i$, then:

**Proposition 2** *(*Incentive compatibility and continuity of best-reply*) Under Assumption 1, $\forall i \in \mathcal{I}$, $\forall s_{-i} \in \mathcal{S}_{-i}$, such that $\tilde{Q}_i(0, s_{-i}) = 0$, for any $\epsilon > 0$, there exists a truthful $\epsilon$-best reply $t_i(s_{-i}) \in \mathcal{T}_i$.*

*In particular, let*

$$G_i(s_{-i}) = \left\{ z \in \bigcap_{l \in r_i} \mathcal{Q}^l : z \leq \tilde{Q}_i(\theta_i'(z), s_{-i}) \ \textit{and} \ \int_0^z \tilde{P}_i(\eta, s_{-i})\, d\eta \leq b_i \right\}.$$

*Then with $v_i = [\sup G_i(s_{-i}) - \epsilon/\theta_i'(0)]^+$ and $w_i = \theta_i'(v_i)$, $t_i = (v_i, w_i) \in \mathcal{T}_i \cap S_i^\epsilon(s_{-i})$.*

*Further, $t_i$ is continuous in $s_{-i}$ on any subset $V_i(\underline{P}, \overline{P}) = \{s_{-i} \in \mathcal{S}_i : \forall z > 0, \overline{P} \geq \tilde{P}_i(z, s_{-i}) \geq \underline{P}\}$, with $\infty > \overline{P} \geq \underline{P} > 0$.*

*In addition, $\tilde{a}_i(t_i; s_{-i}) = v_i$.*

**Proof:** See [10]. $\square$

Figure 1 illustrates the consistent and truthful best reply for a player with a two-hop route.

**Proposition 3** *(*Network Nash equilibrium*) In the network auction game with the PSP rule applied independently at each link, reserve prices $p_0^l > 0, \forall l \in \mathcal{L}$, and players described by (4) and (5), if Assumption 1 holds, then for any $\epsilon > 0$, there exists a consistent and truthful $\epsilon$-Nash equilibrium $s^* \in \mathcal{T}$.*

**Proof:** $\forall s \in \mathcal{T}, i \in \mathcal{I}, \updownarrow \in \mathcal{L}, \ddagger > \imath$, we have $z > 0 = Q_i^l(p_0^l/2, s_{-i}^l)$, which by (7) implies $P_i(z, s_{-i}) \geq p_0/2 = \underline{P}$. Let $\overline{P} = \max_{k \in \mathcal{I}} \theta_k'(0) \vee \max_{l \in \mathcal{L}} p_0^l$. Then, by Proposition 2, $t = (v, w)$ is continuous in $s$ on $\mathcal{T}$. From the last statement of Proposition 2), we have $\tilde{a}_i(t_i; s_{-i}) = v_i$. Therefore $(z_i(s), y_i(s)) \stackrel{def}{=} x_i(t_i(s); s_{-i}) =$

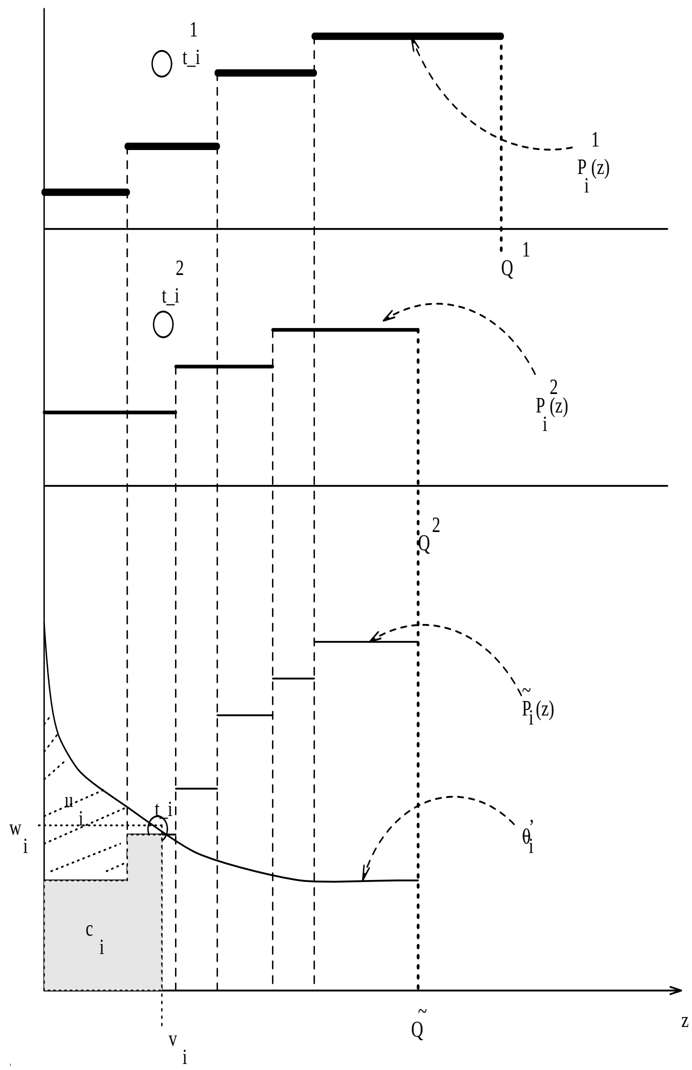


Figure 1: Consistent truthful bid for two hop route

$(1_{r_i}(l) \min_{m \in r_i} v_i^m(s), 1_{r_i}(l) \max_{m \in r_i} \theta_i'(v_i^m))$, is continuous. Let $(q,p) \stackrel{def}{=} s$. Since $s \in \mathcal{T}$, we have $s = (q, \theta'(q))$. By Assumption 1, $\forall i \in \mathcal{I}$, $\theta_i'$ is continuous therefore $z$ can be viewed as a continuous mapping of $\mathcal{Q}^I$ onto itself. By Brouwer's fixed-point theorem (see for example [6]), any continuous mapping of a convex compact set into itself has at least one fixed point, i.e. $\exists q^* = z(q^*) \in [0, Q]^I$. Now with $s^* = (q^*, \theta'(q^*))$, we have $s^* = t(s^*) \in \mathcal{T}$. □

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