FAST RECOVERY ALGORITHMS FOR TIME ENCODED BANDLIMITED SIGNALS

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ABSTRACT

Time encoding is a real-time asynchronous mechanism of mapping amplitude information into a time sequence. We investigate fast algorithms for the recovery of time encoded bandlimited signals and construct an algorithm that has provably low computational complexity. We also devise a fast algorithm that is parameter-insensitive.

1. INTRODUCTION

Time encoding is a real-time asynchronous mechanism of mapping amplitude information into a time sequence. In [4] time encoding and irregular sampling were shown to be largely equivalent modalities of information representation and recovery. For communications applications, however, irregular sampling requires the transmission of both amplitude and sampling time information. Time encoding requires only the transmission of the time sequence. Thus, capacity requirements for time encoded versus irregular sampled bandlimited signals are lower by a factor of two [5].

For the case of irregular sampling, fast algorithms for signal recovery have been extensively studied in the literature (*e.g.*, [2], [3], [6]). While these algorithms are an excellent starting point to search for fast algorithms for recovery of time encoded bandlimited signals, they can not be directly applied to the time encoding case. The reasons are purely technical: the method developed for the irregular sampling case calls for reducing the solution of a linear systems of equations to a Toeplitz matrix inversion. Our method for obtaining a fast algorithm for the time encoding case is based on the observation that the indefinite integral of the same signal can be directly recovered from its amplitude values sampled at instances provided by the time sequence. In the process we find that the complexity of recovering the indefinite integral of an arbitrary bandlimited signal from irregular samples is essentially the same as the complexity of recovering the same signal from its time encoded sequence. In addition, we find that the condition numbers of the pseudo-inverse matrices that arise in both formulations can be chosen to fall in the same range.

This paper is organized as follows. A brief overview of the classical recovery algorithm for irregular sampling followed by a review of an efficient recovery for irregular sampling is presented in section 2. In section 3 we derive a fast recovery algorithm for time encoded bandlimited signals. In section 4, a parameter-insensitive reconstruction algorithm is devised. Simulation results are presented in section 5.

2. FAST RECOVERY ALGORITHM FOR IRREGULAR SAMPLING

2.1. The Classical Recovery Algorithm for Irregular Sampling

Given a set of irregular sampling times (t_k), $k \in \mathbb{Z}$, we shall represent the Ω-bandlimited signal $x = x(t)$, $t \in \mathbb{R}$, as

$$x(t) = \mathbf{g}^T\mathbf{c} = \sum_{k\in\mathbb{Z}} c_k g(t - t_k), \tag{1}$$

where $\mathbf{c} = [c_k]$ is a vector of weights, T denotes the transpose and $\mathbf{g} = [g(t - t_k)]$ with

$$g(t) = \frac{\sin \Omega t}{\pi t} \tag{2}$$

is the vector of t_k-shifted impulse response of an ideal low-pass filter with bandwidth Ω. Also, $\mathbb{R}$ and $\mathbb{Z}$ above denote the set of real numbers and integers, respectively. Denoting by:

$$[\mathbf{q}]_l = x(t_l) \text{ and } [\mathbf{G}]_{l,k} = g(t_l - t_k), \tag{3}$$

for all k, $k \in \mathbb{Z}$, and $l, l \in \mathbb{Z}$, it is easy to see that

$$\mathbf{q} = \mathbf{G}\mathbf{c}. \tag{4}$$

If the average density of the s_k's is at or above the Nyquist rate then, $\mathbf{c}$ can be evaluated as

$$\mathbf{c} = \mathbf{G}^{+}\mathbf{q}, \tag{5}$$

where $\mathbf{G}^{+}$ is the pseudo-inverse (Moore-Penrose) of $\mathbf{G}$. The practical solution, however, is challenging because $\mathbf{G}$ is typically ill-conditioned, and, $\mathbf{q}$, $\mathbf{c}$ and $\mathbf{G}$ are infinite dimensional.

2.2. A Fast Recovery Algorithm for Irregular Sampling

For irregular sampling, let us consider with $T = \pi/\Omega$ and $\alpha^{-1} = (2M+1)T$

$$g(t) = \alpha \sum_{n=-M}^{M} e^{jn\frac{\Omega}{M}t} = \alpha \frac{\sin\left(\frac{2M+1}{2M}\Omega t\right)}{\sin\left(\frac{\Omega t}{2M}\right)} \tag{6}$$

instead of the impulse response defined in (2). When M tends to infinity this function converges to the original impulse response. At the same time, the latter choice of $g(t)$ is both Ω-bandlimited and periodic with a period $2MT$.

In analogy to (1), we define the function $f = f(t), t \in \mathbb{R}$, by

$$f(t) = \alpha \sum_{k\in\mathbb{Z}} c_k \sum_{n=-M}^{M} e^{jn\frac{\Omega}{M}(t-t_k)}. \tag{7}$$

Assuming an appropriate norm, f is an approximation of x as $M \to \infty$ provided that $f(t_l) = x(t_l)$ for all $l, l \in \mathbb{Z}$ [3]. Evaluating the above equality at $t = t_l$ we obtain

$$f(t_l) = \alpha \sum_{k\in\mathbb{Z}} c_k \sum_{n=-M}^{M} e^{jn\frac{\Omega}{M}(t_l-t_k)}, \tag{8}$$

for all $l, l \in \mathbb{Z}$. By denoting $[\mathbf{q}]_l = x(t_l)$ and $[\mathbf{S}]_{m,l} = e^{-jm\frac{\Omega}{M}t_l}$, equation (8) above after multiplication with $\mathbf{S}$ becomes

$$\mathbf{Sq} = \alpha \mathbf{SS}^H\mathbf{Sc}, \tag{9}$$

or

$$\mathbf{d} = \alpha \mathbf{T}^{+}\mathbf{Sq}, \tag{10}$$

where

$$\mathbf{d} = \alpha\mathbf{Sc} \quad \text{and} \quad \mathbf{T} = \alpha\mathbf{SS}^H. \tag{11}$$

For the record

$$[\mathbf{d}]_n = \alpha \sum_{k\in\mathbb{Z}} c_k e^{-jn\frac{\Omega}{M}t_k} \quad \text{and} \quad [\mathbf{T}]_{m,n} = \alpha \sum_{l\in\mathbb{Z}} e^{j(n-m)\frac{\Omega}{M}t_l}. \tag{12}$$

Note that $\mathbf{T}$ is both Toeplitz and Hermitian since $[\mathbf{T}]_{m,n} = [\mathbf{T}]_{m-n}$ and $\mathbf{T}^H = \mathbf{T}$ (the superscript H indicates conjugate transposition). Finally with $[\mathbf{d}]_n = d_n$,

$$f(t) = \sum_{n=-M}^{M} d_n e^{jn\frac{\Omega}{M}t}, \tag{13}$$

In summary, the bandlimited signal x can be approximately represented by equation (13) with the weighting coefficients evaluated following (10). The low complexity of the algorithm (10) is due to the fact that $\mathbf{T}$ is a Hermitian Toeplitz matrix.

3. FAST RECOVERY ALGORITHM FOR TIME ENCODING

3.1. Classical Recovery Algorithm for Time Encoding

In the time encoding case an Ω-bandlimited signal $x = x(t)$, $t \in \mathbb{R}$, is represented as a discrete strictly increasing time sequence (t_k), $k \in \mathbb{Z}$. The time sequence (t_k), $k \in \mathbb{Z}$, is generated using a Time Encoding Machine (TEM) [4]. An example of a TEM is depicted in Figure 1 [4]. It consists of an ideal integrator and a noninverting Schmitt trigger in a feedback arrangement. The output $z(t)$ takes the values b or $-b$ at transition times denoted by t_k. It can be shown [4] that this circuit is described by for all $l, l \in \mathbb{Z}$, by the recursive equation

$$\int_{t_l}^{t_{l+1}} x(u)du = (-1)^k \left[2\kappa\delta - b(t_{k+1} - t_k\right]. \tag{14}$$

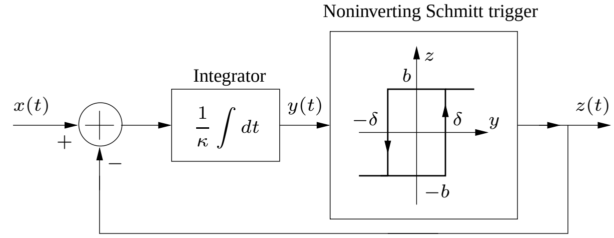

Fig. 1. An Example of a Time Encoding Machine

The recovery of the signal x is achieved via the representation

$$x(t) = \mathbf{g}^T\mathbf{c} = \sum_{k\in\mathbb{Z}} c_k g(t - s_k) \tag{15}$$

with an appropriate set of weights c_k, $k \in \mathbb{Z}$. Denoting by:

$$[\mathbf{q}]_l = \int_{t_l}^{t_{l+1}} x(u)du \text{ and } [\mathbf{G}]_{l,k} = \int_{t_l}^{t_{l+1}} g(u - s_k)du, \tag{16}$$

where $s_k = (t_{k+1} + t_k)/2$, it is easy to see that

$$\mathbf{q} = \mathbf{Gc} \text{ and } \mathbf{c} = \mathbf{G}^{+}\mathbf{q}. \tag{17}$$

3.2. Reformulation of the Classical Recovery Algorithm

If x is a bandlimited function, then so is $\int_{-\infty}^{t} x(u)du$, $t \in \mathbb{R}$, and therefore:

$$\int_{-\infty}^{t} x(u)du = \mathbf{g}^T\mathbf{c} = \sum_{k\in\mathbb{Z}} c_k g(t - t_k), \tag{18}$$

where $\mathbf{g} = [g(t - t_k)]$, $g(t)$ is given by (2) and $\mathbf{c} = [c_k]$ is an appropriate set of coefficients. Since

$$\int_{t_l}^{t_{l+1}} x(u)du = \sum_{k \in \mathbb{Z}} c_k [g(t_{l+1} - t_k) - g(t_l - t_k)], \quad (19)$$

we have

$$\mathbf{q} = \mathbf{PGc}, \quad (20)$$

where $[\mathbf{P}]_{l,k} = \delta_{l+1,k} - \delta_{l,k}$ (using Kronecker's notation) and $[\mathbf{G}]_{l,k} = [g(t_l - t_k)]$ (same as in (3)) and thus

$$\mathbf{c} = \mathbf{G}^{+}\mathbf{P^{-1}q} \quad (21)$$

with

$$[\mathbf{P}^{-1}]_{i,k} = \begin{cases} -1 & \text{if} \quad i \leq k \\ 0 & \text{if} \quad i > k. \end{cases} \quad (22)$$

Remark 1 Note that by multiplying both sides of equation (20) with $t_{l+1} - t_l$, we obtain:

$$(t_{l+1} - t_l)[\mathbf{P^{-1}q}]_l = \sum_{k \in \mathbb{Z}} \underbrace{(t_{l+1} - t_l)\frac{\sin \Omega(t_l - t_k)}{\pi(t_l - t_k)}}_{[\,\cdot\,]_{l,k}} c_k. \quad (23)$$

In the time encoding case,

$$\begin{aligned} [\mathbf{q}]_l &= \sum_{k \in \mathbb{Z}} \int_{t_l}^{t_{l+1}} g(u - s_k)du \; c_k \\ &= \sum_{k \in \mathbb{Z}} (t_{l+1} - t_l)\frac{\sin \Omega(\xi_l - s_k)}{\pi(\xi_l - s_k)} c_k \end{aligned} \quad (24)$$

for some $\xi_l \in [t_{t_l}, t_{l+1}]$. Therefore, the elements of the matrix identified by the lower brace in equation (23) are approximately equal to the elements of the $\mathbf{G}$ matrix for time encoding. This points to the close relationship between the recovery algorithm of a time encoded bandlimited signal x and the recovery of an irregularly sampled integrated signal $\int_{-\infty}^{t} x(u)du$. For both recovery methods the same time sequence is used.

3.3. Fast Recovery Algorithm

Now, (23) can be transformed into a Toeplitz system in the same way as carried out in Sec. 2.2. Replacing the sinc term by the approximation introduced in (6) gives:

$$(t_{l+1} - t_l)[\mathbf{P^{-1}q}]_l = \alpha(t_{l+1} - t_l) \sum_{k \in \mathbb{Z}} c_k \sum_{n=-M}^{M} e^{jn\frac{\Omega}{M}(t_l - t_k)}$$

Denoting by $\mathbf{D} = \text{diag}\,(t_{l+1} - t_l)$, $l \in \mathbb{Z}$, we have in matrix form

$$\mathbf{DP^{-1}q} = \alpha \mathbf{DS}^H \mathbf{Sc}.$$

Multiplying both sides by $\mathbf{S}$ from the left gives:

$$\mathbf{SDP^{-1}q} = \alpha \mathbf{SDS}^H \mathbf{Sc}$$

Equivalently with

$$\mathbf{T} = \alpha \mathbf{SDS}^H, \quad \mathbf{d} = \alpha \mathbf{Sc}, \quad (25)$$

we have

$$\mathbf{d} = \alpha \mathbf{T}^{+}\mathbf{SDP^{-1}q}. \quad (26)$$

The matrix $\mathbf{T}$ is both Toeplitz and Hermitian since

$$[\mathbf{T}]_{n,m} = \alpha \sum_{k \in \mathbb{Z}} (t_{k+1} - t_k) e^{j(m-n)\frac{\Omega}{M} t_k}.$$

Finally, the original signal x is approximated the function f given by

$$f(t) = \frac{j\Omega}{M} \sum_{n=-M}^{M} n d_n e^{jn\frac{\Omega}{M}t}, \quad (27)$$

with the vector $\mathbf{d}$ given by (26).

4. PARAMETER-INSENSITIVE RECONSTRUCTION

The original reconstruction of a time encoded bandlimited function depends on the TEM parameters κ and δ. In [4] the Compensation Principle was used to address the insensitivity of the recovery algorithm with respect to the parameters of the TEM. A simple argument shows, however, that the method employed in [4] does not directly apply here. Therefore, a different technique is needed.

Parameter insensitivity can be achieved if the term $\mathbf{P^{-1}q}$ in equation (25) does not depend on κ and δ. Carrying out $\mathbf{P}^{-1}\mathbf{q}$ by using (22) gives:

$$[\mathbf{P^{-1}q}]_l = -\kappa\delta \left[1 + (-1)^i\right] + b \sum_{k=i}^{L} (-1)^k (t_{k+1} - t_k)$$

where L denotes the size of $\mathbf{G}$ (ideally $L \to \infty$). Therefore, as seen, $\mathbf{P}^{-1}\mathbf{q}$ will be independent of $\kappa\delta$ for odd values of i. If -1 in the last column of every second line of $\mathbf{P}^{-1}$ is changed to zero then each integral is carried out over even number of subintervals. This can be achieved by adding the vector

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{q} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{a}} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{b}^T} \mathbf{q}$$

to $\mathbf{P}^{-1}\mathbf{q}$ and defining

$$\mathbf{p} = \mathbf{P}^{-1}\mathbf{q} + \mathbf{a}\mathbf{b}^T\mathbf{q}, \tag{28}$$

where for illustration purposes and in finite dimensions

$$\mathbf{p} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}.$$

Note that $\mathbf{p}$ can be calculated without using parameters δ and κ.
Since

$$\mathbf{p} = \mathbf{P^{-1}q} + \mathbf{a}\mathbf{b}^T\mathbf{P}\mathbf{P^{-1}q}$$

and $\mathbf{P^{-1}q} = \alpha\mathbf{S}^H\mathbf{Sc}$, we have:

$$\mathbf{p} = \alpha\mathbf{S}^H\mathbf{Sc} + \alpha\mathbf{a}\mathbf{b}^T\mathbf{P}\mathbf{S}^H\mathbf{Sc}.$$

Multiplying both sides above by $\mathbf{SD}$ gives:

$$\mathbf{SDp} = \alpha\mathbf{SDS}^H\mathbf{Sc} + \alpha\mathbf{SDab}^T\mathbf{PS}^H\mathbf{Sc}.$$

Using the matrix $\mathbf{T}$ and vector $\mathbf{d}$ defined in (25) and (26), respectively, and denoting by

$$\mathbf{u} = \alpha\mathbf{SDa}, \quad \mathbf{v}^T = \mathbf{b}^T\mathbf{PS}^H, \tag{29}$$

we have:

$$(\mathbf{T} + \mathbf{u}\mathbf{v}^T)\mathbf{d} = \mathbf{SDp},$$

and therefore

$$\mathbf{d} = (\mathbf{T} + \mathbf{u}\mathbf{v}^T)^{+}\mathbf{SDp}. \tag{30}$$

Finally, employing the result of [1] (page 50, Corollary 3.3.1), the pseudo-inversion in (30) can be determined based on the pseudo-inverse of $\mathbf{T}$ as

$$(\mathbf{T} + \mathbf{u}\mathbf{v}^T)^{+} = \mathbf{T}^{+} - \frac{\mathbf{T}^{+}\mathbf{u}\mathbf{v}^T\mathbf{T}^{+}}{1 + \mathbf{v}^T\mathbf{T}^{+}\mathbf{u}}$$

provided that $1+\mathbf{v}^T\mathbf{T}^{+}\mathbf{u} \neq 0$. Thus, to solve (30) we again have to calculate the pseudo- inverse of a Hermitian Toeplitz matrix and the parameter-insensitivity is also guaranteed. The reconstructed signal is again given by (27).

5. SIMULATION RESULTS

With $\Omega = \pi \times 80$ kHz and the corresponding Nyquist period $T = \pi/\Omega = 12.5\ \mu$s the bandlimited signal $x(t)$ was generated by its Shannon-representation

$$x(t) = \sum_{n=-35}^{65} x(nT)\mathrm{sinc}(\Omega(t - nT)),$$

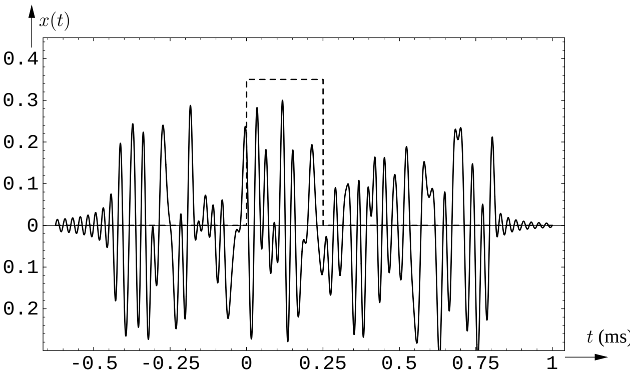

Fig. 2. Overall input signal $x(t)$ and the simulation range (dashed box) for time encoding.

where the samples $x(-35T), \ldots, x(65T)$ were randomly selected in the range $(-0.3, 0.3)$ and sinc$(t) = \sin t/t$ if $t \neq 0$ and sinc$(0) = 1$. This is shown in Fig. 2 in the range $t \in [-50T, 80T]$. The dashed box in the figure shows the simulation range $t \in [0\ \mu\text{s}, 250.38\ \mu\text{s}]$ for the TEM. With parameters $\delta = 7 \times 10^{-6}$, $\kappa = 1/2$, and $c = 0.3$ the TEM simulation produced 35 trigger times $t_0 = 0, t_1, \ldots, t_{35}$. Fig. 3 shows the error signal defined as the difference between the right-hand-side and the left-hand-side of (27) in an even further reduced range $t \in [25.125\ \mu\text{s}, 225.38\ \mu\text{s}]$ to decrease the boundary effects [4].

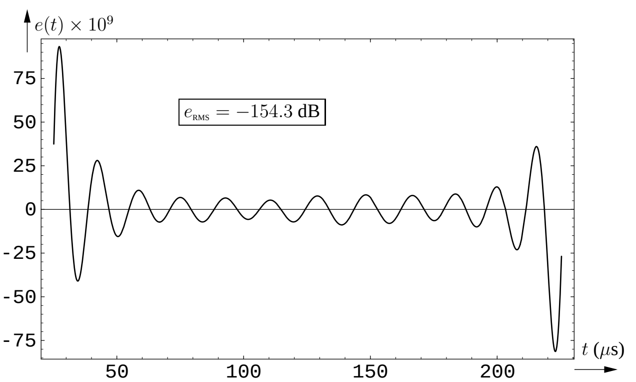

Fig. 3. Error signal using the original formulation in the reduced range to reduce the boundary effects.

Finally, Fig. 4 shows the results for the RMS error and the condition number of $\mathbf{T}$ (based on using $\|\cdot\|_2$) in terms of different values of M by using the new reconstruction technique. It can be seen that as the condition number increases the accuracy of the reconstruction improves.

Fig. 5 shows the simulation results with the original (squares) and the parameter-insensitive reconstruction technique (stars). The error with the original method is the same as that shown in Fig. 4.

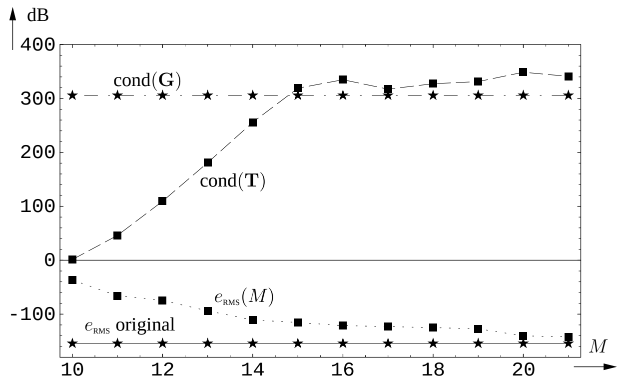

Fig. 4. Simulation results for the condition numbers and RMS error.

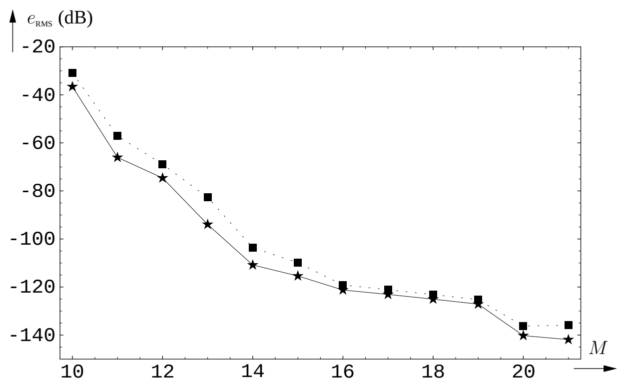

Fig. 5. Simulation results for the RMS error.

Using the same parameters as before we found that increasing M the accuracy improves. The nonzero eigenvalues of $\mathbf{T}$ are approaching to those of $\mathbf{G}$ in (23) and practically zero eigenvalues are introduced if M is increased. In particular, having 27 trigger times the error for $M = 11$ ($2M+1 = 23$), $M = 13$ ($2M+1 = 27$), $M = 15$ ($2M+1 = 31$), the corresponding error turned out to be -111.86 dB, -139.96 dB, -146.28. The accuracy cannot be improved below -154.3 dB corresponding to the case when the original $g(t)$ is used.

6. CONCLUSIONS

We have shown that the indefinite integral of an arbitrary bandlimited signal can be directly recovered from the time encoded sequence associated with the same bandlimited signal. This simple observation enabled us to devise a fast algorithm for signal recovery. We have also demonstrated that our approach can be extended to parameter insensitive signal recovery. Taken together, these results shed further light on the close relationship between irregular sampling and time encoding.

7. REFERENCES

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