EE E6970: Resource Allocation and Networking Games
Prof. A. A. Lazar
Notes for Section 1 (01/27/97)
by Michael E. Kounavis
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Auctions for Resource Sharing

1. Introduction

In real networking environments, where multiple users compete with each other, in order to acquire available resources, centralized algorithms for pricing, control, scheduling etc. are not applicable. The reason why, is because individual users' properties and objectives cannot be estimated and optimally compared in a both scaleable and centralized manner. We consider the use of an auction as a decentralized mechanism for efficiently and fairly sharing resources inside a network. Auction is a decentralized mechanism because prices are not calculated by an a-priori formula, but derive from the users' different valuations and willingness to pay for resources.

In this session the auction rules of Second Price (Vickerey Auction) and Progressive Second Price are being presented and analyzed. Their importance lies on the fact that these rules demonstrate a number of nice properties such as stability, efficiency and fairness. Furthermore, through their analysis, the reader is introduced to the concept of Nash Equilibrium, which is one of the fundamental concepts in Game Theory.

2. Formulation of Auctions for Resource Sharing

Let us consider that a quantity Q of a resource is to be shared among several *players*. Q might represent a portion of the wireless spectrum, a number of TDMA slots etc. *Auction* is a mechanism consisting of players submitting bids, i.e. declaring their desired share of the resource and the price they are willing to pay for it, and the *auctioneer* who allocates shares of the resources based on their bids.

Let I = {1, … , K} be the set of players competing for the resource Q. We define the *k*-player's bid as the vector $s_k = (q_k, p_k) \in S_k \equiv [0, Q] \times [0, \infty)$, where q_k is the amount if resource that the player is biding for, and p_k is the price per unit of resource he is willing to pay. A *bid profile s* is a K×2 array, that contains the bids of the players that participate in the game.

$$s = \begin{pmatrix} s_1 \\ s_2 \\ \cdots \\ s_K \end{pmatrix} = \begin{pmatrix} q_1 & p_1 \\ q_2 & p_2 \\ \cdots & \cdots \\ q_K & p_K \end{pmatrix}$$

We will now introduce the following notation, which is really helpful in order to present all subsequent propositions and lemmata:

$$qs = s\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}q_1\\q_2\\q_3\\q_4\end{pmatrix} \;,\quad ps = s\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}p_1\\p_2\\p_3\\p_4\end{pmatrix} \quad \text{so that} \quad s = (ps, qs)$$

We similarly denote k-player's quantity and price in the following manner :

$$qs_k = s_k\begin{pmatrix}1\\0\end{pmatrix} = (q_k \quad p_k)\begin{pmatrix}1\\0\end{pmatrix} = q_k \;,\quad ps_k = s_k\begin{pmatrix}0\\1\end{pmatrix} = p_k \;,\quad s_k = (ps_k, qs_k)$$

and the opponent bid profile as :

$$s_{-k} = \begin{pmatrix}s_1\\ \dots \\ s_{k-1}\\ s_{k+1}\\ \dots \\ s_K\end{pmatrix}, \quad \text{so that} \quad s = (s_k, s_{-k})$$

The *allocation rule* A(s) is a function that describes the procedure of dividing the resource among players and assigning prices according to their bids. The allocation is done by the auctioneer. A(s) is formally defined as shown below :

$$\text{A:} \quad S^K \rightarrow S^K, \quad \text{where} \quad S^K = \prod_{k=1}^{K} S \quad \text{and} \quad S = [0, Q] \times [0, \infty)$$

The k-th row of the matrix A(s) is the allocation to player k, which consists of the amount of resource the player actually gets and the price he pays for it.

$$A_k(s) = (qA_k(s) \quad pA_k(s))$$

We say that an allocation rule A(s) is *feasible* if both the following conditions are satisfied:

- The sum of all the amounts allocated to players does not exceed the available resource Q.

$$\sum_{k=1}^{K} qA_k(s) \le Q$$

- Every player is never given more quantity than he asks for, and the price he has to pay does not exceed the one declared through his bid.

$$A(s) \le s$$

where the operator less-or-equal (≤) denotes element by element comparison.

Each player's objective is described by a function $u_k(s)$ which is called *utility function*, or simply *utility*. This function takes as argument a bid profile and a returns a value that expresses how profitable is the corresponding allocation to a particular player. In the following sections we will use the definition for $u_k(s)$, shown below.

$$u_k(s) = \theta_k q A_k(s) - q A_k(s) p A_k(s)$$

We assume that player k has a valuation $\theta_k \geq 0$ for each unit of the resource he gets, so the total value of his allocation is $\theta_k q A_k(s)$. His utility is thus defined as the difference between the value and the cost of the acquired quantity.

Finally we consider an *auction game* is defined by an allocation rule, a set of utility functions and an amount of available resource:

$$\left(Q, u_1(s), u_2(s), \ldots, u_K(s), A(s)\right)$$

3. Second Price Auction

The Second Price Auction (also called Vicrey Auction) is an auction game designed, in such a way that it enforces players to bid at their own valuation of the resource. This property of the game is called *incentive compatibility* and its importance lies on the fact that every player's utility is maximized, exactly when he bids truthfully.

In the Second Price Auction we assume that the resource is non-divisible. This means that $q_k = Q$ for all k. The bid profile thus has the following form:

$$S = \begin{pmatrix} Q & p_1 \\ Q & p_2 \\ \ldots & \ldots \\ Q & p_K \end{pmatrix}$$

Let us consider that players are ordered in such a way that :

$$0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_K$$

According to the *Second Price allocation rule,* the player who bids at the highest price gets the resource, and he pays the second highest price. Now, let us consider each player's utility $u_k(s)$. As we mentioned in the previous section the utility function for each player is defined as the difference between the actual value of the quantity he gets and its cost. Thus, for the Second Price auction game:

$$u_k = \theta_K - \max_{l \neq k} p_k, \quad \text{if} \quad p_k > \max_{l \neq k} p_l$$

or 0 otherwise.

Now we will state and prove the following lemma:

Lemma 1: *(incentive compatibility)*

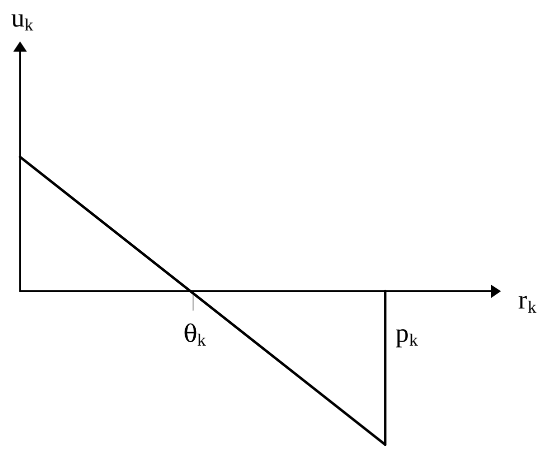
For each player k, the strategy of bidding at his own valuation $p_k = \theta_k$ weakly dominates other strategies, i.e.:

$$u_k\left(p_1, p_2, \ldots, \theta_k, \ldots, p_k\right) \geq u_k\left(p_{1,} p_2, \ldots, p_k, \ldots, p_K\right)$$

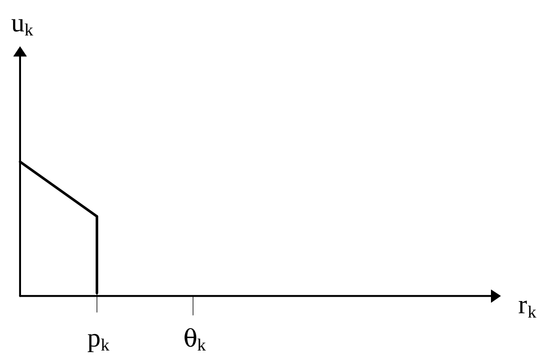
Proof:

Let $r_k = \max_{l \neq k} p_k$. The proof for the lemma presented above derives directly from the graphical representations of u_k as a function of r_k.

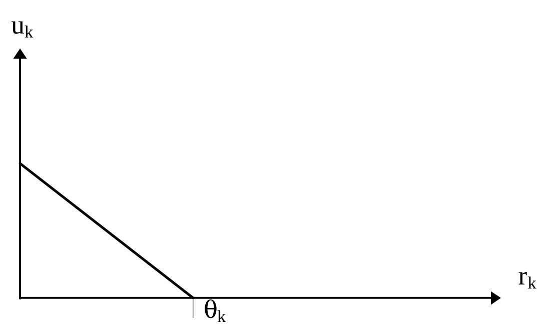
Situation 1: $\theta_k < p_k$



Situation 2: $\theta_k > p_k$



Situation 3: $\theta_k = p_k$



For $r_k \leq \theta_k$, we observe that u_k decreases linearly, until it reaches zero, in situations 1 and 3, whereas, in situation 2, it decreases rapidly to zero at a particular value (p_k) before the valuation θ_k. This happens because for $r_k > p_k$, the player is being allocated no quantity and therefore his utility is zero.

For $r_k \geq \theta_k$, the utility is zero in situations 2 and 3 and negative for some values of r_k, in situation 1. This happens because the player bids at prices higher than his valuation and therefore has to pay more than the value of the resource, in case he wins the auction game.

Comparing situations 1,2, and 3 we observe that for every r_k the following property holds:

$$u_k\left(p_1, p_2, \ldots, \theta_k, \ldots, p_k\right) \geq u_k\left(p_{1,} p_2, \ldots, p_k, \ldots, p_K\right)$$

4. Progressive Second Price Auction

The Progressive Second Price (PSP) auction game, is the generalization of the Vicrey Auction for the case of divisible resources. As we will show latter, the PSP game is incentive compatible and can result in equilibrium points which are both *fair* and *efficient*.

The difference between the bid profile in the Progressive Second Price and Second Price games is that quantities may be lower than Q:

$$s = \begin{pmatrix} q_1 & p_1 \\ \ldots & \ldots \\ q_K & p_K \end{pmatrix}$$

The PSP allocation rule is described below :

$$qA_k(s) = \min\left(qs_k, \left[Q - \sum_{l: ps_l > ps_k} qA_l(s)\right]^+\right)$$

$$pA_k(s) = \sum_{l \neq k} ps_l \frac{qA_l\left(0; s_{-k}\right) - qA_l\left(s_k; s_{-k}\right)}{\sum_{m \neq k}\left[qA_m\left(0; s_{-k}\right) - qA_m\left(s_k; s_{-k}\right)\right]}$$

The first equation presented above tells us that a player can get the quantity he asks for, provided that the resource which is not given to higher price players is sufficient for him. Otherwise, he is being allocated the leftover amount, which may be nothing at all in the worst case. A player is more likely to get the quantity he wants if he bids at relatively high prices.

Now, the second equation reveals the exclusion-compensation principle, which is the fundamental design guideline for the PSP game. In fact, player k has to pay the average unit price that would have been paid by other players if there were no player k in the game. The denominator in the expression shown above is in fact the quantity $qA_k(s)$, and the summation takes actually place for all l: $ps_l < ps_k$, because higher price players are not really affected by player k.

It is worth noticing that the unit price $pA_k(s)$ increases with $qA_k(s)$ in a way similar to the income tax rate. Let's assume a scenario where we have K players and apply PSP allocation rule without initially considering player k in the game. Let m be the lowest clearing opponent. It is obvious that in order for player k to be given some quantity, he has to bid at a price $ps_k \geq ps_m$. Now, let $qA_k = 0$. While qA_k increases the first few units will be taken away from player m, and they will cost ps_m. After player k takes all the quantity qA_m, he will start taking units of resource from the next lowest clearing opponent, m' and these units will be more expensive than the previous ones because $ps_{m'} > ps_m$. So, the more quantity a player asks for, the higher unit price he is likely to pay for this amount.

We can see that, in the case where $qs_1 = qs_2 = \dots = qs_K = Q$, the PSP allocation rule is the same as the one of the Vicrey auction game. The quantity is being allocated to the highest price player, and the price he has to pay is the second highest price, because if the player does not participate in the game the whole resource is being given to the second highest player.

We can also develop some intuition to see why the Progressive Second Price auction is indeed incentive compatible. Generally, as we said before, if a player bids at higher prices he is more likely to get the amount he wants. And so, his utility is being increased. But how high can his bidding price be? Well, the exclusion-compensation principle discourages players from bidding at prices higher than their own valuation, because for some units of resource, they would have to pay more than their actual value.

In the next section we will formally prove the incentive compatibility property.

5. Incentive Compatibility

Before presenting the incentive compatibility property, we need to show some useful lemmata:

Lemma 2:

If $s_k \le s_{k'}$ then $qA_k(s_k;s_{-k}) \le qA_k(s_{k'};s_{-k})$, and
if $s_{-k} \le s_{-k'}$ then $qA_k(s_k;s_{-k}) \ge qA_k(s_k;s_{-k'})$

These equations tell us that a player is being allocated less or equal quantity if he decreases his bid, and greater or equal quantity if the opponent profile decreases.

Proof :

If $s_k \le s_{k'}$ then $qs_k \le qs_{k'}$ and $ps_k \le ps_{k'}$. So,

$$ps_k \le ps_{k'} \Rightarrow \sum_{l:ps_l>ps_k} qA_l(s) \ge \sum_{l:ps_l>ps_{k'}} qA_l(s) \Rightarrow$$
$$[Q - \sum_{l:ps_l>ps_k} qA_l(s)]^+ \le [Q - \sum_{l:ps_l>ps_{k'}} qA_l(s)]^+ \Rightarrow$$
$$\min(qs_k, [Q - \sum_{l:ps_l>ps_k} qA_l(s)]^+) \le \min(qs_{k'}, [Q - \sum_{l:ps_l>ps_{k'}} qA_l(s)]^+) \Rightarrow$$
$$qA_k(s_k;s_{-k}) \le qA_k(s_{k'};s_{-k})$$

Similarly, if $s_{-k} \le s_{-k'}$ then

$$\sum_{l:ps_l>ps_k} qA_l(s) \le \sum_{l':ps_{l'}>ps_k} qA_{l'}(s) \Rightarrow$$
$$[Q - \sum_{l:ps_l>ps_k} qA_l(s)]^+ \ge [Q - \sum_{l':ps_{l'}>ps_k} qA_{l'}(s)]^+ \Rightarrow$$
$$\min(qs_k, [Q - \sum_{l:ps_l>ps_k} qA_l(s)]^+) \ge \min(qs_k, [Q - \sum_{l':ps_{l'}>ps_k} qA_{l'}(s)]^+) \Rightarrow$$
$$qA_k(s_k;s_{-k}) \ge qA_k(s_k;s_{-k'})$$

Lemma 3:

For all opponent profiles $s_{-k} \in S^{K-1}$:

$$u_k(s_{k'};s_{-k}) - u_k(s_k;s_{-k}) = \sum_{l \neq k} (\theta_k - ps_l)\left[qA_l(s_k;s_{-k}) - qA_l(s_{k'};s_{-k})\right]$$

Proof:

As we mentioned in the previous section the quantity allocated to player k is equal to the sum of all additional quantities that his opponents acquire, if the player does not participate in the game:

$$qA_k(s_k;s_{-k}) = \sum_{l \neq k}\left[qA_l(0;s_{-k}) - qA_l(s_k;s_{-k})\right]$$

Using this equation, player k's utility can be written as:

$$\begin{aligned} u_k(s_k;s_{-k}) &= \sum_{l \neq k}\left[qA_l(0;s_{-k}) - qA_l(s_k;s_{-k})\right]\theta_k - \sum_{l \neq k}\left[qA_l(0;s_{-k}) - qA_l(s_k;s_{-k})\right]ps_k \\ &= \sum_{l \neq k}\left[qA_l(0;s_{-k}) - qA_l(s_k;s_{-k})\right]\theta_k - \sum_{l \neq k} ps_l\left[qA_l(0;s_{-k}) - qA_l(s_k;s_{-k})\right] \\ &= \sum_{l \neq k}(\theta_k - ps_l)\left[qA_l(0;s_{-k}) - qA_l(s_k;s_{-k})\right] \end{aligned}$$

Similarly :

$$u_k(s_{k'};s_{-k}) = \sum_{l \neq k}(\theta_k - ps_l)\left[qA_l(0;s_{-k}) - qA_l(s_{k'};s_{-k})\right]$$

Subtracting the last equation from the previous one, we have that:

$$u_k(s_{k'};s_{-k}) - u_k(s_k;s_{-k}) = \sum_{l \neq k} (\theta_k - ps_l)\left[qA_l(s_k;s_{-k}) - qA_l(s_{k'};s_{-k})\right]$$

Lemma 4: *(incentive compatibility)*

Given an opponent profile s_{-k}:

$$u_k((qs_k,\theta_k);s_{-k}) \geq u_k((qs_k,ps_k);s_{-k})$$

Proof:

Using Lemma 3, we can write that:

$$u_k((qs_k,\theta_k);s_{-k}) = u_k((qs_k,ps_k);s_{-k}) + \sum_{l \neq k}(\theta_k - ps_l)\left[qA_l((qs_k,ps_k);s_{-k}) - qA_l((qs_k,\theta_k);s_{-k})\right]$$

So, we only have to show that :

$$\sum_{l \neq k}(\theta_k - ps_l)\left[qA_l((qs_k,ps_k);s_{-k}) - qA_l((qs_k,\theta_k);s_{-k})\right] \geq 0$$

Situation 1

We assume that $ps_k \le \theta_k$. Player k, belongs to the set of player l's opponents. Therefore using Lemma 1 we have that:

$$qA_l\left((qs_k, ps_k); s_{-k}\right) - qA_l\left((qs_k, \theta_k); s_{-k}\right) \ge 0$$

Now, for all l: $ps_l \le \theta_k$,

$$(\theta_k - ps_l)[qA_l\left((qs_k, ps_k); s_{-k}\right) - qA_l\left((qs_k, \theta_k); s_{-k}\right)] \ge 0 \Rightarrow$$

$$\sum_{l: ps_l \le \theta_k} (\theta_k - ps_l)[qA_l\left((qs_k, ps_k); s_{-k}\right) - qA_l\left((qs_k, \theta_k); s_{-k}\right)] \ge 0$$

For all the players that $ps_l > \theta_k$, their allocation is not affected by player k, who bids at lower prices. Therefore:

$$qA_l\left((qs_k, ps_k); s_{-k}\right) - qA_l\left((qs_k, \theta_k); s_{-k}\right) = 0$$

So, we can write that:

$$\sum_{l \ne k} (\theta_k - ps_l)[qA_l\left((qs_k, ps_k); s_{-k}\right) - qA_l\left((qs_k, \theta_k); s_{-k}\right)] \ge 0$$

Situation 2

Now, we consider that $ps_k \ge \theta_k$. Using Lemma 1 we have that:

$$qA_l\left((qs_k, ps_k); s_{-k}\right) - qA_l\left((qs_k, \theta_k); s_{-k}\right) \le 0$$

For all l: $ps_l \ge \theta_k$,

$$(\theta_k - ps_l)[qA_l\left((qs_k, ps_k); s_{-k}\right) - qA_l\left((qs_k, \theta_k); s_{-k}\right)] \ge 0 \Rightarrow$$

$$\sum_{l: ps_l \ge \theta_k} (\theta_k - ps_l)[qA_l\left((qs_k, ps_k); s_{-k}\right) - qA_l\left((qs_k, \theta_k); s_{-k}\right)] \ge 0$$

For all the players that $ps_l < \theta_k$, their allocation is not affected by player k, who bids at higher prices and does not alter his quantity.

$$qA_l\left((qs_k, ps_k); s_{-k}\right) - qA_l\left((qs_k, \theta_k); s_{-k}\right) = 0$$

So, we have that:

$$\sum_{l \ne k} (\theta_k - ps_l)[qA_l\left((qs_k, ps_k); s_{-k}\right) - qA_l\left((qs_k, \theta_k); s_{-k}\right)] \ge 0$$

hence the incentive compatibility property is proven.

6. Nash Equilibrium

So far we explored the really attractive property of incentive compatibility. What it tells us is that given a particular quantity qs_k the truthful bid $s_k = (\theta_k, ps_k)$ is always optimal. However, the optimal reply that derives from the incentive compatibility lemma, presupposes that the bidding quantity qs_k is known. What we are interested in, is to explore whether a game like the PSP auction can reach an operating point, in which it is not beneficial for every user to deviate from it. Such an operating point is called *Nash equilibrium.*

We consider that every player is constrained by a *budget* $b_k \geq 0$, so that his bid must lie in the set:

$$S_k(s_{-k}) = \{s_k \in S: qA_k(s_k; s_{-k})pA(s_k; s_{-k}) \leq b_k\}$$

We define the set of best replies to a profile s_{-k} of opponent bids as the set:

$$S_k^*(s_{-k}) = \{s_k \in S_k(s_{-k}): u_k(s_k; s_{-k}) \geq u_k(s'_k; s_{-k}), \forall s'_k \in S_k(s_{-k})\}$$

Now let $S^*(s) = \prod_k S_k^*(s_{-k})$. A Nash Equilibrium is defined as a fixed point of S^*, i.e. a profile s^* such that $s_k^* \in S_k^*(s_{-k}^*), \forall k$.

Now, an interesting question is, if we can find among Nash Equilibria, solutions which are of a simplified form as indicated by the incentive compatibility property.

Let us consider the unconstrained set of player k's truthful bids:

$$T_k = \{s_k \in S: ps_k = \theta_k\}$$

We also define: $T = \prod_k T_k$, $T_k(s_{-k}) = T_k \cap S_k(s_{-k})$ the set of constrained truthful bids, and $R_k = T_k \cap S_k^*(s_{-k})$, the set of truthful optimal replies.

Lemma5: *(existence of truthful best reply)*

For every player $k \in K$, and for every opponent profile $s_{-k} \in \prod_{l \neq k} S$,

$$R(s_{-k}) \neq \emptyset$$

Therefore, by setting $ps_k = \theta_k, \forall k$ we can reduce the parameters to the problem and still obtain feasible best replies. Thus a proper truthful sub-game is formed, where the strategy space is $T \subset S$, the feasible sets are $T_k(s_{-k}) \subset S_k(s_{-k})$, and the best replies are $R(s) \subset S^*(s)$. A fixed point of R in T is a fixed point of S^* in S. Thus an equilibrium of the sub-game is an equilibrium of the game.

Proposition 1:

In the PSP auction game defined in the previous sections, there exists a Nash Equilibrium point $s^* \in T$.

7. Equilibrium properties: Fairness

One of the most intuitively appealing definitions of a *fair* allocation is the *envy-free* definition. We say that an allocation is envy-free if every player k, does not "envy" his opponents l. This means that if player k is assigned quantity $qA_l(s)$ at price $pA_l(s)$, his utility does not increase, for every l≠k.

We formally define the envy-free allocation rule as the one for which:

$$u_k\left(A_k(s)\right) \geq u_k\left(A_l(s)\right), \forall k, l \in K$$

From the definition of the envy-free allocation, we can see that a rule A(s) is not fair, if some players are being assigned more quantities at lower prices, in relation to others. If for a particular player k, there exists a player l for which $qA_l(s) \geq qA_k(s)$ and $pA_l(s) \leq pA_k(s)$, then:

$$u_k\left(A_k(s)\right) = qA_k(s)\left[\theta_k - pA_k(s)\right] \leq qA_k\left[\theta_k - pA_l(s)\right] \leq qA_l(s)\left[\theta_k - pA_l(s)\right] = u_k\left(A_l(s)\right)$$

Proposition 2:

Let us consider that players are ordered in such a way that :

$$0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_K$$

If there exists $m \in K$, such that

$$\sum_{l>m} \frac{b_l}{\theta_m} + \frac{b_m}{\theta_{m-1}} \geq Q \geq \sum_{l>m} \frac{b_l}{\theta_m}$$

and, for every k>m,

$$\sum_{l:l>m} \frac{b_l}{\theta_m} + \frac{\theta_k - \theta_m}{\theta_k - \theta_{m-1}} \frac{b_k}{\theta_m} \geq Q$$

then, there exists an equilibrium point $s^* \in T$ such that the allocation $A(s^*)$ is fair.

Intuition:

Let us consider that player m fixes his bid at $s_m = (Q, \theta_m)$. Then the other players have an equilibrium point s^*_{-m}. By Lemma 4 (incentive compatibility) s_m is an optimal reply, provided that it is feasible. The first of the equations presented above exactly assures that s_m is feasible. If this is the case, then m is the lowest clearing opponent. In the equilibrium point players k with k>m are being assigned quantities b_k / θ_m at price θ_m, whereas player m is being assigned the leftover amount $Q - \sum_{l:l>m} \frac{b_l}{\theta_m}$ at price θ_{m-1}.

This allocation is indeed envy-free, because players k >m cannot envy each other since they pay the same unit price and they all get the maximum quantity for their budget, and none of them envies player m. The latter assertion derives from the second equation shown above.

$$u_k\left(A_k(s)\right) \geq u_k\left(A_m(s)\right) \Leftrightarrow$$

$$\frac{b_k}{\theta_m}[\theta_k - \theta_m] \geq \left(Q - \sum_{l:l>m} \frac{b_l}{\theta_m} \right)[\theta_k - \theta_{m-1}] \Leftrightarrow$$

$$\sum_{l:l>m} \frac{b_l}{\theta_m} + \frac{\theta_k - \theta_m}{\theta_k - \theta_{m-1}} \frac{b_k}{\theta_m} \geq Q$$

8. References

[1] A. A. Lazar and N. Semret "Auctions for Network Resource Sharing", CTR Technical Report CU/CTR/TR 468-97-02, Columbia University February 11, 1997.

[2] D. Fudenberg and J.Tirole "Game Theory" MIT Press, 1991.