# Complexity Distortion Theory

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*Abstract* — **We investigate the efficiency of lossy algorithmic representations of information and show that "Complexity Distortion" is asymptotically equivalent to Rate Distortion for stationary ergodic sources.**

## I. Introduction

The concept of efficiently representing information dates back to the late 40's with the pioneering work of C. E. Shannon. Since then, compression has been a continuously growing research field with numerous fundamental contributions to communication system design and implemention. Several diverse techniques have emerged recently, including fractals, model-based coding, and sophisticated programmable designs that allow flexibility and extensibility via downloadable code [2]. Most of these techniques cannot be analyzed within traditional theoretical frameworks. We propose a new theory, called Complexity Distortion Theory, which uses complexities or length of lossy descriptions to provide a much broader and unifying perspective on media representation. The key component of this theory is the substitution of Shannon's classical communication system model by Chaitin's model [1], where the decoder is a universal Turing machine and the codewords are programs for such a computer. In this paper, we focus on optimal representations using the Algorithmic-Kolmogorov complexity and show that, for stationary ergodic sources and despite its deterministic nature, this measure of information predicts asymptotically the same results as the classical probabilistic notion of information[1].

## II. Optimal Quantization and the $C(D)$ Function

Let $K(x_n)$ and $K(x_n \mid y_n)$ be respectively the unconditional and conditional Algorithmic-Kolmogorov complexities of a binary string $x_n$ of length n, without or with prior knowledge of another binary string $y_n$ [1]. Let $\mathcal{D}$ be a distortion measure according to any valid distance metric $d(\cdot,\cdot)$. On a given object $x_n$, with D as a constraint, we introduce distortion in order to minimize the complexity of the resulting object $y_n$. If we have more than one object $y_n$ with the same optimal complexity, we select the closest to $x_n$. If many optimal objects are equidistant to $x_n$, we make the selection arbitrarily. We clearly define a function from the set of source objects to the set of distorted objects. We denote this function $\mathcal{D}$.

**Definition 1** *The complexity distortion function is:*

$$C(D) = \lim_{n\to\infty} E[\frac{K(\mathcal{D}(x_n))}{n}] = \lim_{n\to\infty} E[\frac{K(x_n) - K(x_n \mid \mathcal{D}x_n)}{n}]$$

[1]Similar results are given in [3] but in a less constructive way.

## III. Equivalence between $R(D)$ and $C(D)$

**Theorem 1** *For ergodic sources,* $R(D) = C(D) = \lim_{n\to\infty} \frac{K(x_n)-K(x_n|\mathcal{D}(x_n))}{n}$*, with probability 1,* $R(D)$ *being the classical rate distortion function.*

The proof of this result can be divided in two main parts. First, we define two programs yielding upper bounds for $\lim_{n\to\infty} \frac{K(x_n)}{n}$ and $\lim_{n\to\infty} \frac{K(x_n|\mathcal{D}(x_n))}{n}$. In the unconditional case, from all the possible sequences with the same type or empirical distribution, output the $i$-th corresponding to the one we want to describe. In the conditional case, form all the possible sequences with the same type and the same image by $\mathcal{D}$, output the $i'$-th corresponding to the one we want to describe. The lengths of these programs are the size in bits of the indices $i$ and $i'$, up to an additive constant. We relate them to the entropy rate of the source process using the types. In the second part of the proof, we use the incompressibility property of most of the infinite binary string, in an algorithmic sense, to prove that the set of sequences with descriptions shorter than the one proposed here has measure 0. This completes the proof.

## IV. Unification of Coding Techniques

Using the Church-Turing thesis stating that everything that can be done algorithmically can be done with Turing machine, we obtain the foundation of the Complexity Distortion Theory, a unified perspective on media representation that includes both modern and traditional signal representation techniques and provides the utmost flexibility and universality in terms of capabilities of the receiver. Sophisticated techniques such as model-based coding or synthetic-natural hybrid coding can be naturally analyzed using this new framework, where traditional information theoretic techniques would simply fail. In addition, resource bounds on computational time and space at the decoder can be naturally introduced, in contrast with classical Rate Distortion Theory. Finally, Complexity Distortion Theory allows for the existence of downloadable decoding components, which is one of the features of the forthcoming MPEG-4 standard [2].

## References

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