1 Image Histogram Stretching

1.1 Histogram of Gray-image

Fig. 1 shows the original gray image “coin.bmp”, which has a spatial resolution of 350 × 232 pixels with 8 bits per pixel. To generate the histogram of a gray image which is equivalent
to an 1D color image, the approach is the same as that employed for each single color channel of a 3D color image. Please see the solution of Homework 1 for more details. Fig. 2 shows the histograms of the original gray image “coin.bmp” in the cases of \( n = 128 \) and \( n = 256 \) bins. It is seen from Fig. 2 that the distribution of the pixel values is far from an even distribution over the entire region. In other words, most of the pixels fall in a relatively small region.

### 1.2 Histogram Stretching

The piecewise mapping function for the stretching operation can be analytically derived as follows. Denote \( \beta_1, \beta_2 \) and \( \beta_3 \) as the intercepts for the three lines corresponding to the three regions \([0, \text{stint}_\text{min}]), [\text{stint}_\text{min}, \text{stint}_\text{max}], \) and \((\text{stint}_\text{max}, M]\), respectively. Denote \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) as the slopes for the three lines corresponding to the three regions \([0, \text{stint}_\text{min}]), [\text{stint}_\text{min}, \text{stint}_\text{max}], \) and \((\text{stint}_\text{max}, M]\), respectively. The slopes can be simply written as

\[
\alpha_1 = \frac{\text{stout}_\text{min}}{\text{stint}_\text{min}}, \tag{1}
\]
\[
\alpha_2 = \frac{\text{stout}_\text{max} - \text{stout}_\text{min}}{\text{stint}_\text{max} - \text{stint}_\text{min}}, \tag{2}
\]
\[
\alpha_3 = \frac{M - \text{stout}_\text{max}}{M - \text{stint}_\text{max}}. \tag{3}
\]

Note that the intercepts satisfy the following equations in geometry:

\[
\beta_1 = 0, \\
\frac{\text{stint}_\text{max}}{\text{stint}_\text{min}} = \frac{\text{stout}_\text{max} - \beta_2}{\text{stout}_\text{min} - \beta_2}, \\
\frac{M}{\text{stint}_\text{max}} = \frac{M - \beta_3}{\text{stout}_\text{max} - \beta_3},
\]

from which the intercepts can be derived

\[
\beta_1 = 0, \tag{4}
\]
\[
\beta_2 = -\frac{\text{stint}_\text{min} \text{stout}_\text{max} - \text{stint}_\text{max} \text{stout}_\text{min}}{\text{stint}_\text{max} - \text{stint}_\text{min}}, \tag{5}
\]
\[
\beta_3 = M \frac{\text{stout}_\text{max} - \text{stint}_\text{max}}{M - \text{stint}_\text{max}}. \tag{6}
\]
Figure 2: Histograms of the original gray image “coin.bmp”: $n = 128$ and $n = 256$. 
Hence, the piecewise stretching operations can be analytically expressed by

\[ \text{Out} = \alpha_i \text{In} + \beta_i, \quad i = 1, 2, 3, \]  

(7)

where \( \alpha_i \) and \( \beta_i \) are defined in (1)–(3) and (4)–(6), respectively; the four parameters \( \text{stin}_{\text{min}}, \text{stin}_{\text{max}}, \text{stout}_{\text{min}}, \) and \( \text{stout}_{\text{max}} \) will be further detailed in Section 1.3.

### 1.3 Parameters of Histogram Stretching

The parameters of the histogram input, \( \text{stin}_{\text{min}} \) and \( \text{stin}_{\text{max}} \), depends on the histogram of the original gray image “coin.bmp” which is shown in the bottom of Fig. 2. It is seen from Fig. 2 that in the case of \( n = 256 \), the gray values of most of the pixels fall in the region \([100, 230]\). Meanwhile, to transform the histogram into the one as spreadly distributed as possible, the parameters of the histogram output, \( \text{stout}_{\text{min}} \) and \( \text{stout}_{\text{max}} \), should be as small as possible and as large as possible, respectively. In the simulations to be demonstrated in Section 1.4, for \( n = 256 \) and \( M = 256 \), we choose the values of the above parameters as follows:

- **input parameters**: \( \text{stin}_{\text{min}} = 100, \text{stin}_{\text{max}} = 230 \);  
- **output parameters**: \( \text{stout}_{\text{min}} = 10, \text{stout}_{\text{max}} = 250 \).  

(8)  

(9)

Note that, to gain more insight into the role of the output parameters, we choose another set of values as a comparison:

\( \text{stout}_{\text{min}} = 50, \text{stout}_{\text{max}} = 240. \)  

(10)

### 1.4 Simulation Results

Fig. 3 shows the histogram comparison between the original image and the stretched image using (9). It is seen from Fig. 3 that the histogram is more spread over the entire region after the stretching operation (7). Similarly, Fig. 4 shows the same comparison between the
original image and the stretched image using (10), and the similar conclusion can be drawn as from Fig. 3. Moreover, Fig. 5 shows the comparison between the stretched histograms using different output parameters (9) and (10). It is seen from Fig. 5 that the larger the stout_{max} is and the smaller the stout_{min} is, the histogram after stretching operation (7) is more spread. Moreover, Fig. 6 shows the subjective performance comparison between the original image and the stretched ones using (9) and (10). It is seen from Fig. 6 that the images after the stretching operation (7) can offer a better subjective performance than the original one; and consistent with what Fig. 5 has demonstrated, (9) can achieve a better subjective performance than (10).
Figure 3: Comparison between the original histogram and the stretched histogram of the gray image “coin.bmp”: $M = 256$; $\text{stin}_{\text{min}} = 100$ and $\text{stin}_{\text{max}} = 230$ (i.e., (8)); $\text{stout}_{\text{min}} = 10$ and $\text{stout}_{\text{max}} = 250$ (i.e., (9)).
Figure 4: Comparison between the original histogram and the stretched histogram of the gray image “coin.bmp”: $M = 256$; $\text{stin}_{\text{min}} = 100$ and $\text{stin}_{\text{max}} = 230$ (i.e., (8)); $\text{stout}_{\text{min}} = 50$ and $\text{stout}_{\text{max}} = 240$ (i.e., (10)).
Figure 5: Comparison between the stretched histograms using different output parameters (9) and (10).
Figure 6: Subjective performance comparison between the original image and the stretched images: $M = 256$; the bottom image—the original one; the upper-left image—the stretched one using (9); the upper-right image—the stretched one using (10).
2 Histogram Transformation

Denote $U$ and $V$ as two random variables representing the input and the output of the histogram transformation, respectively. Denote $p_U(u)$ and $F_U(u) = \int_{-\infty}^{u} p_U(u)du$ as the probability density function (pdf) and the cumulative probability density functions (cdf) of $U$, respectively. Similarly, $p_V(v)$ and $F_V(v) = \int_{-\infty}^{v} p_V(v)dv$ denote the pdf and the cdf of $V$, respectively. The histogram transformation has the following constraint

$$F_U(u) = F_V(v).$$

(11)

Obviously, the mapping function $v = T(u)$ corresponding to the histogram transformation (11) depends on the specific distributions of both $U$ and $V$.

2.1 Mapping Function: Cube-root Hyperbolically Distributed $V$

Assume that $V$ is a random variable with a cube-root hyperbolic distribution, i.e.,

$$p_V(v) = \begin{cases} \frac{v^{-2/3}}{3(v_{\max}^{1/3} - v_{\min}^{1/3})}, & \text{if } v_{\min} \leq v \leq v_{\max}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the cdf of $V$ can be further written as

$$F_V(v) = \int_{-\infty}^{v} p_V(v)dv = \begin{cases} 1, & v > v_{\max}, \\ \int_{v_{\min}}^{v} p_V(v)dv, & v_{\min} \leq v \leq v_{\max}, \\ 0, & v < v_{\min}. \end{cases}$$

(12)

Note that in (12), only the region $v \in [v_{\min}, v_{\max}]$ is treated in the histogram transformation, and the corresponding cdf of $V$ is given by

$$F_V(v) = \frac{v^{1/3} - v_{\min}^{1/3}}{v_{\max}^{1/3} - v_{\min}^{1/3}}, \quad v \in [v_{\min}, v_{\max}].$$

Inserting (12) into (11), we then have the following mapping function corresponding to the cube-root hyperbolic distribution of $V$

$$F_U(u) = F_V(v) = \frac{v^{1/3} - v_{\min}^{1/3}}{v_{\max}^{1/3} - v_{\min}^{1/3}}.$$
\[ v = T(u) = [(v_{\text{max}}^{1/3} - v_{\text{min}}^{1/3})F_U(u) + v_{\text{max}}^{1/3}]^3. \]  

(13)

2.2 Mapping Function: Uniformly Distributed $U$

It is observed from (13) that $u$ is involved in the mapping function $v = T(u)$ by $F_U(u)$ which has no closed-form expression as long as the pdf of $U$ is unknown. Now assume that $U$ is a random variable with a uniform distribution over $[0, 1]$, i.e.,

\[
p_U(u) = \begin{cases} 
1, & 0 \leq u \leq 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Then the cdf of $U$ can be further written as

\[
F_U(u) = \int_{-\infty}^{u} p_U(u)du = \begin{cases} 
1, & u > 1, \\
\int_{0}^{u} p_U(u)du = u, & 0 \leq u \leq 1, \\
0, & u < 0.
\end{cases}  \tag{14}
\]

Similarly as revealed in (12), only the region $u \in [0, 1]$ in (14) is treated in the histogram transformation. Inserting (14) into the mapping function (13) obtained in Section 2.1, we find the mapping function corresponding to the cube-root hyperbolic distribution of $V$ and the uniform distribution of $U$:

\[ v = T(u) = [(v_{\text{max}}^{1/3} - v_{\text{min}}^{1/3})u + v_{\text{max}}^{1/3}]^3. \]  

(15)

Fig. 7 shows the plot of the mapping function $v = T(u)$ in (15) where $v_{\text{min}} = 0$ and $v_{\text{max}} = 1$, i.e., $v = T(u) = u^3$.  

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Figure 7: Mapping function $v = T(u)$ in (15): $v_{\text{min}} = 0$ and $v_{\text{max}} = 1$. 